

CLASSIFICATION OF ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP IN CHARACTERISTIC 2

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ABSTRACT. We classify supersingular and classical Enriques surfaces with finite automorphism group in characteristic 2 into 8 types according to their dual graphs of all (-2) -curves (nonsingular rational curves). We give examples of these Enriques surfaces together with their canonical coverings. It follows that the classification of all Enriques surfaces with finite automorphism group in any characteristics has been finished.

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1. INTRODUCTION

We work over an algebraically closed field k of characteristic 2. The main purpose of this paper is to give a classification of supersingular and classical Enriques surfaces with finite automorphism group in characteristic 2.

Recall that, over the complex numbers, Nikulin [26] proposed a classification of Enriques surfaces with finite automorphism group in terms of the periods, and the second author [15] classified and gave constructions of all such Enriques surfaces, geometrically. There are seven types I, II, \dots , VII of such Enriques surfaces. The Enriques surfaces of

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type I and II form irreducible 1-dimensional families, and each of the remaining types consists of a unique Enriques surface. The first two types contain exactly twelve (-2) -curves (i.e. nonsingular rational curves), while the remaining five types contain exactly twenty (-2) -curves. We call the dual graphs of all (-2) -curves on the Enriques surface of type K the dual graph of type K ($K = \text{I, II, } \dots, \text{VII}$). We remark that if an Enriques surface has the dual graph of this type, then its automorphism group is finite.

In positive characteristics, the question of a classification of Enriques surfaces with finite automorphism group has been raised. In particular, the case of characteristic 2 is most interesting. In the paper [1], Bombieri and Mumford classified Enriques surfaces in characteristic 2 into three classes, namely singular, classical and supersingular Enriques surfaces. As in the case of characteristic 0 or $p > 2$, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : \tilde{X} \rightarrow X$, which is a separable $\mathbf{Z}/2\mathbf{Z}$ -cover, a purely inseparable μ_2 - or α_2 -cover according to X being singular, classical or supersingular. The surface \tilde{X} might have singularities and it might even be non-normal, but it is $K3$ -like in the sense that its dualizing sheaf is trivial. Recently, Liedtke [20] showed that the moduli space of Enriques surfaces with a polarization of degree 4 has two 10-dimensional irreducible components. One component (resp. the other component) consists of singular (resp. classical) Enriques surfaces. The intersection of the two components parametrizes supersingular Enriques surfaces. On the other hand, Ekedahl and Shepherd-Barron [7] studied special Enriques surfaces called "exceptional" and Salomonsson [30] gave equations of such Enriques surfaces. We remark that some of them have a finite group of automorphisms.

Very recently, the first and the second authors [13] determined the existence or non-existence of Enriques surfaces in characteristic 2 whose dual graphs of all (-2) -curves are of type I, II, \dots , or VII which is given as in the following Table 1:

Type	I	II	III	IV	V	VI	VII
singular	○	○	×	×	×	○	×
classical	×	×	×	×	×	×	○
supersingular	×	×	×	×	×	×	○

TABLE 1

In Table 1, ○ means the existence and × means the non-existence of an Enriques surface with the dual graph of type I, \dots , VII. All examples in the Table 1 are given explicitly. On the other hand, the third author [21] gives a classification of Enriques surfaces with finite automorphism group in characteristic $p > 2$ by using the method given by the second author over the complex numbers. The classification is the same as over the complex numbers except that in the case of $p = 5$, Enriques surfaces of type VI and VII do not appear, and that if $p = 3$, the ones of type V and VI do not appear (see Martin [21]). Moreover, a similar method shows that the classification of singular Enriques surfaces

with finite automorphism group in characteristic 2 is given as in the above Table 1. Thus, the classification problem remains only for the classical and supersingular cases. Now, we state the main results of this paper.

Theorem 1.1. *Let X be a supersingular Enriques surface in characteristic 2.*

- (A) *X has a finite group of automorphisms if and only if the dual graph of all (-2) -curves on X is one of the graphs in Table 2 (A).*
- (B) *All cases exist. More precisely, we construct families of these surfaces whose automorphism groups and dimensions are given in Table 2 (B).*



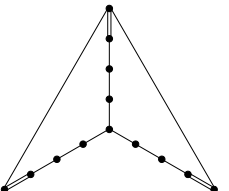

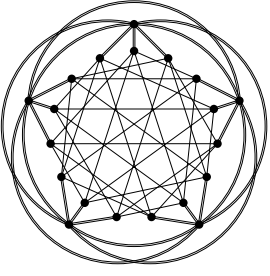
(A) Classification		(B) Examples		
Type	Dual Graph of (-2) -curves	$\text{Aut}(X)$	$\text{Aut}_{ct}(X)$	dim
\tilde{E}_8		$\mathbf{Z}/11\mathbf{Z}$	$\mathbf{Z}/11\mathbf{Z}$	0
$\tilde{E}_7 + \tilde{A}_1$		$\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/14\mathbf{Z}$	$\{1\}$ or $\mathbf{Z}/7\mathbf{Z}$	1
$\tilde{E}_6 + \tilde{A}_2$		$\mathbf{Z}/5\mathbf{Z} \times \mathfrak{S}_3$	$\mathbf{Z}/5\mathbf{Z}$	0
\tilde{D}_8		Q_8	Q_8	1
VII		\mathfrak{S}_5	$\{1\}$	0

TABLE 2

Theorem 1.2. *Let X be a classical Enriques surface in characteristic 2.*

- (A) *X has a finite group of automorphisms if and only if the dual graph of all (-2) -curves on X is one of the graphs in Table 3 (A).*

(B) *All cases exist. More precisely, we construct families of these surfaces whose automorphism groups and dimensions are given in Table 3 (B).*

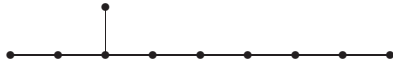


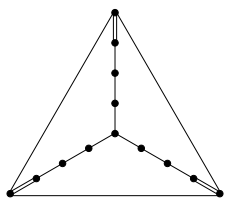


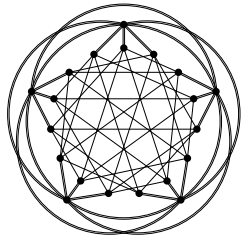
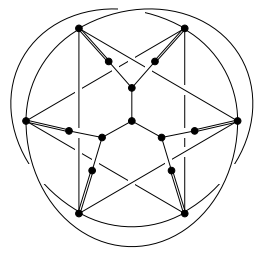
(A) Classification		(B) Examples		
Type	Dual Graph of (-2) -curves	$\text{Aut}(X)$	$\text{Aut}_{nt}(X)$	dim
\tilde{E}_8		$\{1\}$	$\{1\}$	1
$\tilde{E}_7 + \tilde{A}_1$		$\mathbf{Z}/2\mathbf{Z}$	$\{1\}$	2
$\tilde{E}_7 + \tilde{A}_1$		$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	1
$\tilde{E}_6 + \tilde{A}_2$		\mathfrak{S}_3	$\{1\}$	1
\tilde{D}_8		$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2
$\tilde{D}_4 + \tilde{D}_4$		$(\mathbf{Z}/2\mathbf{Z})^3$	$(\mathbf{Z}/2\mathbf{Z})^2$	2
VII		\mathfrak{S}_5	$\{1\}$	1
VIII		\mathfrak{S}_4	$\{1\}$	1

TABLE 3

In Theorems 1.1 and 1.2, $\text{Aut}(X)$, $\text{Aut}_{ct}(X)$ or $\text{Aut}_{nt}(X)$ is the automorphism group of X , the cohomologically trivial automorphism group or the numerically trivial automorphism group (see Definition 2.3), respectively, \mathfrak{S}_n is the symmetric group of degree n and Q_8 is the quaternion group of order 8. The examples of supersingular Enriques surfaces of type $\tilde{E}_7 + \tilde{A}_1$ form a 1-dimensional family, but some of their automorphism groups jump up.

Remark 1.3. We remark that the following families in Theorems 1.1 (B) and 1.2 (B) are non-isotrivial: $\tilde{E}_7 + \tilde{A}_1$ supersingular, $\tilde{E}_6 + \tilde{A}_2$ classical, VII classical, and VIII. The family of $\tilde{E}_7 + \tilde{A}_1$ classical surfaces with simple III fiber and the family of type $\tilde{D}_4 + \tilde{D}_4$ contain an at least 1-dimensional, non-isotrivial family. The authors do not know the existence of other examples, that is, the problem of determining the moduli space of such Enriques surfaces is still open.

Over the complex numbers, numerically and cohomologically trivial automorphism groups are completely classified into three types whose orders are 2 or 4 (Mukai and Namikawa [23], Mukai [24], and also see Kondō [15, Theorem 1.7]). On the other hand, the classification of such automorphisms in characteristic 2 is not known. Theorems 1.1 (B) and 1.2 (B) give new examples of such automorphisms in characteristic 2 (that is, the cases of \tilde{E}_8 supersingular, $\tilde{E}_7 + \tilde{A}_1$ supersingular, $\tilde{E}_6 + \tilde{A}_2$ supersingular, \tilde{D}_8 supersingular and $\tilde{D}_4 + \tilde{D}_4$ classical) which are counterexamples to Dolgachev's claim [5, Theorem 4].

Corollary 1.4. *For $G \in \{Q_8, \mathbf{Z}/11\mathbf{Z}, \mathbf{Z}/7\mathbf{Z}, \mathbf{Z}/5\mathbf{Z}\}$, there exists a supersingular Enriques surface X with $\text{Aut}_{ct}(X) = G$. Moreover, there is a classical Enriques surface X in characteristic 2 with $\text{Aut}_{nt}(X) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.*

Note that only the dual graph of type VII appears over the complex numbers. Moreover, the Enriques surface with the dual graph of type VII is unique over the complex numbers, whereas our example in characteristic 2 is a 1-dimensional and non-isotrivial family of classical and supersingular Enriques surfaces with such dual graph (see Theorem 6.1). The canonical cover of any Enriques surface of type VII has 12 rational double points of type A_1 and its minimal resolution is the unique supersingular $K3$ surface with Artin invariant 1. The canonical covers of the other Enriques surfaces in Theorems 1.1 and 1.2 are non-normal rational surfaces.

The dual graphs of type \tilde{E}_8 , $\tilde{E}_7 + \tilde{A}_1$, \tilde{D}_8 and $\tilde{D}_4 + \tilde{D}_4$ appeared in Cossec and Dolgachev [3], Dolgachev and Liedtke [6] and the first three are called "extra special". However, their existence was not known. Also, Enriques surfaces of type \tilde{E}_8 , $\tilde{E}_7 + \tilde{A}_1$, $\tilde{E}_6 + \tilde{A}_2$ are called "exceptional" and were studied deeply by Ekedahl and Shepherd-Barron [7] and Salomonsson [30] from a different point of view.

In the following, we summarize genus one fibrations on each of the above Enriques surfaces (for the notation, see the subsection 2.4, and Propositions 2.7 and 2.8). We indicate that it is either elliptic or quasi-elliptic after the type of singular fibers.

- Type \tilde{E}_8 : (2II^*) (quasi-elliptic);

- Type $\tilde{E}_7 + \tilde{A}_1$ supersingular : $(2\text{III}^*, \text{III})$ (quasi-elliptic), (II^*) (quasi-elliptic);
classical-case 1: $(2\text{III}^*, \text{III})$ (quasi-elliptic), (II^*) (quasi-elliptic);
classical-case 2: $(2\text{III}^*, 2\text{III})$ (quasi-elliptic), (II^*) (quasi-elliptic);
- Type $\tilde{E}_6 + \tilde{A}_2$,
supersingular: $(2\text{IV}^*, \text{IV})$ (elliptic), $(\text{III}^*, 2\text{III})$ (quasi-elliptic);
classical : $(2\text{IV}^*, \text{I}_3, \text{I}_1)$ (elliptic), $(\text{III}^*, 2\text{III})$ (quasi-elliptic);
- Type \tilde{D}_8 :
supersingular: (2I_4^*) (quasi-elliptic), (2II^*) (elliptic), (II^*) (elliptic);
classical: (2I_4^*) (quasi-elliptic), $(2\text{II}^*, \text{I}_1)$ (elliptic), $(\text{II}^*, \text{I}_1)$ (elliptic);
- Type $\tilde{D}_4 + \tilde{D}_4$: $(2\text{I}_0^*, 2\text{I}_0^*)$ (quasi-elliptic), (I_4^*) (elliptic), (2I_4^*) (elliptic);
- Type VII: $(\text{I}_9, \text{I}_1, \text{I}_1, \text{I}_1)$ (elliptic), $(\text{I}_8, 2\text{III})$ (elliptic), $(\text{I}_5, \text{I}_5, \text{I}_1, \text{I}_1)$ (elliptic),
 $(\text{I}_6, 2\text{IV}, \text{I}_2)$ (elliptic);
- Type VIII: $(2\text{I}_1^*, \text{I}_4)$ (elliptic), $(\text{I}_2^*, 2\text{III}, 2\text{III})$ (quasi-elliptic), $(\text{IV}^*, \text{I}_3, \text{I}_1)$ (elliptic).

In case of a classical or supersingular Enriques surface X , there exists a non-zero regular global 1-form η on X . The divisorial part of the set of zeros of η is called the biconductrix and the half of the biconductrix the conductrix. By definition, the canonical cover $\pi : \tilde{X} \rightarrow X$ has a singularity at $P \in Y$ if and only if η vanishes at $\pi(P)$. Ekedahl and Shepherd-Barron [7] classified possible conductrices of elliptic and quasi-elliptic fibrations on classical and supersingular Enriques surfaces.

The outline of the proof of Theorems 1.1 and 1.2 is as follows. First, recall that any Enriques surface X admits a genus one fibration $\pi : X \rightarrow \mathbf{P}^1$, and any genus one fibration on an Enriques surface has a double fiber. Let $J(\pi) : J(X) \rightarrow \mathbf{P}^1$ be the Jacobian fibration associated with π . Then the Mordell-Weil group of $J(\pi)$ acts on X effectively as automorphisms. Now, assume that the automorphism group $\text{Aut}(X)$ is finite. Then, for any genus one fibration π on X , the Mordell-Weil rank of its Jacobian fibration is 0. We will prove that the possible dual graphs of (-2) -curves on X are nothing but those given in Theorems 1.1 (A) and 1.2 (A), by using the condition of Mordell-Weil rank mentioned as above and Ekedahl and Shepherd-Barron's classification of conductrices (Theorem 12.13). Then it follows from a result by Vinberg [31] that $\text{Aut}(X)$ is in fact finite for each Enriques surface X with one of these dual graphs of (-2) -curves (Proposition 2.12).

On the other hand, for each dual graph Γ in Theorems 1.1 (A) and 1.2 (A), we will construct Enriques surfaces with Γ as the dual graph of (-2) -curves (Sections 5–11). To do this, we look at a subdiagram Γ_0 of Γ which is the dual graph of reducible fibers of a special genus one fibration. Here, a genus one fibration is called special if the fibration has a (-2) -curve as a 2-section. We first consider a rational genus one fibration $g : R \rightarrow \mathbf{P}^1$ whose dual graph of reducible fibers is Γ_0 , and we take the Frobenius base change $f : \tilde{R} \rightarrow \mathbf{P}^1$ of g . Then, we give a rational vector field D on \tilde{R} (in Section 3, we give a method to find suitable vector fields). The vector field D might have isolated singularities and hence we take a resolution of singularities, that is, after blowing-ups of \tilde{R} we get a nonsingular

surface Y such that the induced vector field denoted by the same symbol D has no isolated singularities. Then, the quotient surface Y^D of Y by D is nonsingular and the minimal model X of Y^D is the desired Enriques surface.

Finally, we give a remark on how to calculate the automorphism group $\text{Aut}(X)$, which is isomorphic to a subgroup of the symmetry group of the dual graph Γ up to numerically trivial automorphisms. In cases $\tilde{E}_6 + \tilde{A}_2$ (supersingular), \tilde{E}_8 (supersingular and classical), $\tilde{E}_7 + \tilde{A}_1$ (supersingular), \tilde{D}_8 (supersingular and classical) and $\tilde{D}_4 + \tilde{D}_4$, we can not determine the numerically trivial automorphisms from their dual graphs of (-2) -curves geometrically. In these cases, we first find an equation of a surface birationally equivalent to X , and then we reduce the problem to the calculation of the automorphism group of this surface (see Section 4).

From Section 3 to 12, unless mentioned otherwise, all our Enriques surfaces are classical or supersingular.

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2. PRELIMINARIES

2.1. Vector fields. Let k be an algebraically closed field of characteristic $p > 0$, and let S be a nonsingular complete algebraic surface defined over k . We denote by K_S a canonical divisor of S . A rational vector field D on S is said to be p -closed if there exists a rational function f on S such that $D^p = fD$. A vector field D for which $D^p = 0$ is called of additive type, while that for which $D^p = D$ is called of multiplicative type. Let $\{U_i = \text{Spec } A_i\}$ be an affine open covering of S . We set $A_i^D = \{D(\alpha) = 0 \mid \alpha \in A_i\}$. The affine varieties $\{U_i^D = \text{Spec } A_i^D\}$ glue together to define a normal quotient surface S^D .

Now, we assume that D is p -closed. Then, the natural morphism $\pi : S \rightarrow S^D$ is a purely inseparable morphism of degree p . If the affine open covering $\{U_i\}$ of S is fine enough, then taking local coordinates x_i, y_i on U_i , we see that there exist $g_i, h_i \in A_i$ and a rational function f_i such that the divisors defined by $g_i = 0$ and by $h_i = 0$ have no common divisor, and such that

$$D = f_i \left(g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.$$

By Rudakov and Shafarevich [29, Section 1], divisors (f_i) on U_i give a global divisor (D) on S , and zero-cycles defined by the ideal (g_i, h_i) on U_i give a global zero cycle $\langle D \rangle$ on S . A point contained in the support of $\langle D \rangle$ is called an isolated singular point of D . If D

has no isolated singular point, D is said to be divisorial. Rudakov and Shafarevich [29, Theorem 1, Corollary] showed that S^D is nonsingular if $\langle D \rangle = 0$, i.e. D is divisorial. When S^D is nonsingular, they also showed a canonical divisor formula

$$(2.1) \quad K_S \sim \pi^* K_{S^D} + (p-1)(D),$$

where \sim means linear equivalence. As for the Euler number $c_2(S)$ of S , we have a formula

$$(2.2) \quad c_2(S) = \deg \langle D \rangle - K_S \cdot (D) - (D)^2$$

(cf. Katsura and Takeda [14, Proposition 2.1]).

Now we consider an irreducible curve C on S and we set $C' = \pi(C)$. Take an affine open set U_i above such that $C \cap U_i$ is non-empty. The curve C is said to be integral with respect to the vector field D if $g_i \frac{\partial}{\partial x_i} + h_i \frac{\partial}{\partial y_i}$ is tangent to C at a general point of $C \cap U_i$. Then, Rudakov-Shafarevich [29, Proposition 1] showed the following proposition:

Proposition 2.1. (i) *If C is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.*

(ii) *If C is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.*

2.2. Enriques surfaces in characteristic 2. In characteristic 2, a minimal algebraic surface with numerically trivial canonical divisor is called an Enriques surface if the second Betti number is equal to 10. Such surfaces X are divided into three classes (for details, see Bombieri and Mumford [1, Section 3]):

- (i) K_X is not linearly equivalent to zero and $2K_X \sim 0$. Such an Enriques surface is called a classical Enriques surface.
- (ii) $K_X \sim 0$, $H^1(X, \mathcal{O}_X) \cong k$ and the Frobenius map acts on $H^1(X, \mathcal{O}_X)$ bijectively. Such an Enriques surface is called a singular Enriques surface.
- (iii) $K_X \sim 0$, $H^1(X, \mathcal{O}_X) \cong k$ and the Frobenius map is the zero map on $H^1(X, \mathcal{O}_X)$. Such an Enriques surface is called a supersingular Enriques surface.

It is known that Pic_X^τ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ if X is classical, μ_2 if X is singular or α_2 if X is supersingular (Bombieri-Mumford [1, Theorem 2]). As in the case of characteristic 0 or $p > 2$, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : \tilde{X} \rightarrow X$, which is a separable $\mathbf{Z}/2\mathbf{Z}$ -cover, a purely inseparable μ_2 - or α_2 -cover according to X being singular, classical or supersingular. The surface \tilde{X} might have singularities and it might even be non-normal (see Proposition 2.14), but it is $K3$ -like in the sense that its dualizing sheaf is trivial. Note that Ekedahl and Shepherd-Barron [7] use "unipotent" Enriques surfaces for supersingular and classical ones.

2.3. (-2) -curves. Let X be an Enriques surface and let $\text{Num}(X)$ be the quotient of the Néron-Severi group $\text{NS}(X)$ of X by torsion. Then $\text{Num}(X)$ together with the intersection product is an even unimodular lattice of signature $(1, 9)$ (Illusie [8, Corollary 7.3.7], Cossec and Dolgachev [3, Chap. II, Theorem 2.5.1]), and hence is isomorphic to $U \oplus E_8$.

We denote by $O(\text{Num}(X))$ the orthogonal group of $\text{Num}(X)$. The set

$$\{x \in \text{Num}(X) \otimes \mathbf{R} : \langle x, x \rangle > 0\}$$

has two connected components. Denote by $P(X)$ the connected component containing an ample class of X . For $\delta \in \text{Num}(X)$ with $\delta^2 = -2$, we define an isometry s_δ of $\text{Num}(X)$ by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta, \quad x \in \text{Num}(X),$$

which is nothing but the reflection with respect to the hyperplane perpendicular to δ . The isometry s_δ is called the reflection associated with δ . We call a nonsingular rational curve on an Enriques surface or a $K3$ surface a (-2) -curve. For a (-2) -curve E on an Enriques surface X , we identify E with its class in $\text{Num}(X)$. Let $W(X)$ be the subgroup of $O(\text{Num}(X))$ generated by reflections associated with all (-2) -curves on X . Then $P(X)$ is divided into chambers each of which is a fundamental domain with respect to the action of $W(X)$ on $P(X)$. There exists a unique chamber containing an ample class which is nothing but the closure of the ample cone $D(X)$ of X . It is known that the natural map

$$(2.3) \quad \rho_n : \text{Aut}(X) \rightarrow O(\text{Num}(X))$$

has a finite kernel. Since the image $\text{Im}(\rho_n)$ preserves the ample cone, we see $\text{Im}(\rho_n) \cap W(X) = \{1\}$. Therefore $\text{Aut}(X)$ is finite if the index $[O(\text{Num}(X)) : W(X)]$ is finite. Thus we have the following Proposition (see Dolgachev [4, Proposition 3.2]).

Proposition 2.2. *If $W(X)$ is of finite index in $O(\text{Num}(X))$, then $\text{Aut}(X)$ is finite.*

Over the field of complex numbers, the converse of Proposition 2.2 holds by using the Torelli type theorem for Enriques surfaces (Dolgachev [4, Theorem 3.3]).

Definition 2.3. Denote by $\text{Aut}_{nt}(X)$ the kernel of the map ρ_n given by (2.3). Similarly denote by $\text{Aut}_{ct}(X)$ the kernel of the map

$$(2.4) \quad \rho_c : \text{Aut}(X) \rightarrow O(\text{NS}(X)).$$

A non-trivial automorphism is called cohomologically or numerically trivial if it is contained in $\text{Aut}_{ct}(X)$ or $\text{Aut}_{nt}(X)$, respectively. If S is not classical, then $\text{NS}(X) = \text{Num}(X)$ and hence $\text{Aut}_{ct}(X) = \text{Aut}_{nt}(X)$.

2.4. Genus one fibrations. We recall some facts on an elliptic or a quasi-elliptic fibrations on Enriques surfaces. For simplicity, we call an elliptic or a quasi-elliptic fibration a genus one fibration.

Proposition 2.4. (Bombieri and Mumford [1, Theorem 3]) *Every Enriques surface has a genus one fibration.*

Proposition 2.5. (Dolgachev and Liedtke [6, Theorem 4.8.3])

Let $f : X \rightarrow \mathbf{P}^1$ be a genus one fibration on an Enriques surface X in characteristic 2. Then the following hold.

- (i) If X is classical, then f has two tame double fibers, each is either an ordinary elliptic curve or a singular fiber of additive type.
- (ii) If X is singular, then f has one wild double fiber which is a smooth ordinary elliptic curve or a singular fiber of multiplicative type.
- (iii) If X is supersingular, then f has one wild double fiber which is a supersingular elliptic curve or a singular fiber of additive type.

Lemma 2.6. *Let $f : X \rightarrow \mathbf{P}^1$ be an isotrivial genus one fibration on an Enriques surface in characteristic 2. Let F be a double fiber of f such that the underlying reduced fiber F_{red} is an elliptic curve. Then F_{red} has j -invariant 0 if and only if the generic fiber of f also has j -invariant 0.*

Proof. We can assume that the general fiber of f is an elliptic curve. Since f is isotrivial, it becomes trivial after passing to a finite cover of \mathbf{P}^1 . Hence, F is isogeneous to the generic fiber of f . Since having j -invariant 0 is equivalent to being supersingular in characteristic 2 and being supersingular is an isogeny-invariant, we get the result. \square

We use the symbols I_n ($n \geq 1$), I_n^* ($n \geq 0$), II, III, IV, II^* , III^* , IV^* of singular fibers of an elliptic or a quasi-elliptic fibration in the sense of Kodaira. The dual graph of (-2) -curves on a singular fiber of type I_n ($n \geq 2$), I_n^* ($n \geq 0$), III, IV, II^* , III^* or IV^* is an extended Dynkin diagram \tilde{A}_{n-1} , \tilde{D}_{n+4} , \tilde{A}_1 , \tilde{A}_2 , \tilde{E}_8 , \tilde{E}_7 or \tilde{E}_6 , respectively. For a double singular fiber of type F , we write $2F$. Let $f : S \rightarrow \mathbf{P}^1$ be a genus one fibration on a surface S . If, for example, f has a double singular fiber of type III and a singular fiber of type IV^* , then it is said that f has singular fibers $(2\text{III}, \text{IV}^*)$. If f has a section and its Mordell-Weil group is torsion, then f is called extremal. We use the following classifications of extremal rational elliptic and rational quasi-elliptic fibrations.

Proposition 2.7. (Lang [17], [18]) *The following are the singular fibers of extremal elliptic fibrations on rational surfaces:*

$$\begin{aligned} &(\text{II}^*), (\text{II}^*, I_1), (\text{III}^*, I_2), (\text{IV}^*, \text{IV}), (\text{IV}^*, I_3, I_1), (I_4^*), (I_1^*, I_4), \\ &(I_9, I_1, I_1, I_1), (I_8, \text{III}), (I_6, \text{IV}, I_2), (I_5, I_5, I_1, I_1), (I_3, I_3, I_3, I_3). \end{aligned}$$

Proposition 2.8. (Ito [10]) *The following are the singular fibers of quasi-elliptic fibrations on rational surfaces:*

$$\begin{aligned} &(\text{II}^*), (\text{III}^*, \text{III}), (I_4^*), (I_2^*, \text{III}, \text{III}), (I_0^*, I_0^*), \\ &(I_0^*, \text{III}, \text{III}, \text{III}, \text{III}), (\text{III}, \text{III}, \text{III}, \text{III}, \text{III}, \text{III}, \text{III}, \text{III}). \end{aligned}$$

Remark 2.9. Any quasi-elliptic fibration on a rational surface is extremal.

Consider a genus one fibration on an Enriques surface $\pi : X \rightarrow \mathbf{P}^1$. Then the Mordell-Weil group of the Jacobian of π acts on X effectively as automorphisms. This implies the following Proposition.

Proposition 2.10. (Dolgachev [4, §4]) *Assume that the automorphism group of an Enriques surface X is finite. Then any genus one fibration on X is extremal.*

Let X be an Enriques surface. A genus one fibration $f : X \rightarrow \mathbf{P}^1$ is called special if there exists a (-2) -curve R with $R \cdot f^{-1}(P) = 2$ ($P \in \mathbf{P}^1$), that is, f has a (-2) -curve as a 2-section. In this case, R is called a special 2-section. The following result is due to Cossec [2] in which he assumed the characteristic $p \neq 2$, but the assertion for $p = 2$ holds, too.

Proposition 2.11. (Lang [16, II, Theorem A3], Dolgachev and Liedtke [6, Theorem 5.3.4]) *Assume that an Enriques surface X contains a (-2) -curve. Then there exists a special genus one fibration on X .*

2.5. Vinberg's criterion. Let X be an Enriques surface. We recall Vinberg's criterion which guarantees that a group generated by a finite number of reflections is of finite index in $O(\text{Num}(X))$.

Let Δ be a finite set of (-2) -vectors in $\text{Num}(X)$. Let Γ be the graph of Δ , that is, Δ is the set of vertices of Γ and two vertices δ and δ' are joined by m -tuple lines if $\langle \delta, \delta' \rangle = m$. We assume that the cone

$$K(\Gamma) = \{x \in \text{Num}(X) \otimes \mathbf{R} : \langle x, \delta_i \rangle \geq 0, \delta_i \in \Delta\}$$

is a strictly convex cone. Such Γ is called non-degenerate. A connected parabolic subdiagram Γ' in Γ is a Dynkin diagram of type \tilde{A}_m , \tilde{D}_n or \tilde{E}_k (see Vinberg [31, p. 345, Table 2]). If the number of vertices of Γ' is $r + 1$, then r is called the rank of Γ' . A disjoint union of connected parabolic subdiagrams is called a parabolic subdiagram of Γ . We denote by $\tilde{K}_1 \oplus \cdots \oplus \tilde{K}_s$ a parabolic subdiagram which is a disjoint union of connected parabolic subdiagrams of type $\tilde{K}_1, \dots, \tilde{K}_s$, where K_i is A_m , D_n or E_k . The rank of a parabolic subdiagram is the sum of the ranks of its connected components. Note that the dual graph of singular fibers of a genus one fibration on X gives a parabolic subdiagram. We denote by $W(\Gamma)$ the subgroup of $O(\text{Num}(X))$ generated by reflections associated with $\delta \in \Gamma$.

Proposition 2.12. (Vinberg [31, Theorem 2.3]) *Let Δ be a set of (-2) -vectors in $\text{Num}(X)$ and let Γ be the graph of Δ . Assume that Δ is a finite set, Γ is non-degenerate and Γ contains no m -tuple lines with $m \geq 3$. Then $W(\Gamma)$ is of finite index in $O(\text{Num}(X))$ if and only if every connected parabolic subdiagram of Γ is a connected component of some parabolic subdiagram in Γ of rank 8 (= the maximal one).*

Proposition 2.13. (Namikawa [25, Proposition 6.9]) *Let Δ be a finite set of (-2) -curves on an Enriques surface X and let Γ be the graph of Δ . Assume that $W(\Gamma)$ is of finite index in $O(\text{Num}(X))$. Then Δ is the set of all (-2) -curves on X .*

2.6. Conductrix. Let X be a classical or supersingular Enriques surface. Then it is known that there exists a global regular 1-form η on X . The canonical cover $\pi : \tilde{X} \rightarrow X$ has a singularity at $P \in \tilde{X}$ if and only if η vanishes at $\pi(P)$. Since $c_2(X) = 12$, η always vanishes somewhere, and hence \tilde{X} is singular. The divisorial part B of the zero set of η is called the bi-conductrix of X . The divisor B is of the form $2A$ where A is a divisor called the conductrix of X .

Proposition 2.14. (Ekedahl and Shepherd-Barron [7, Proposition 0.5], Dolgachev and Liedtke [6, Proposition 1.3.8]) *Let X be a classical or supersingular Enriques surface and A its conductrix. Assume $A \neq 0$. Then A is 1-connected. Moreover $A^2 = -2$ and the normalization of the canonical cover has either four rational double points of type A_1 as singularities or one rational double point of type D_4 .*

In the paper [7], Ekedahl and Shepherd-Barron gave possibilities of the conductrices for quasi-elliptic and elliptic fibrations in characteristic 2. In Section 12, we will use their classification of the conductrices ([7, Theorems 2.2, 3.1]). For simplicity, we say an A_1 -singularity or a D_4 -singularity for a rational double point of type A_1 or of type D_4 respectively. Also we will use the symbol nA_1 for n rational double points of type A_1 .

3. CONSTRUCTION OF VECTOR FIELDS

In this section, we explain two methods to construct a candidate of a vector field D on an algebraic surface Y such that the quotient surface Y^D becomes an Enriques surface.

3.1. Enriques surfaces with an elliptic pencil. Let $f : Y \rightarrow \mathbf{P}^1$ be an elliptic surface with a section. Assume that Y is either a $K3$ surface or a rational surface. Then, the generic fiber is an elliptic curve E over the field $k(t)$ with one variable t . Therefore, there exists a non-zero regular vector field δ on E which we can regard as a non-zero rational vector field on Y . Taking a suitable vector field $g(t)\frac{\partial}{\partial t}$ and a suitable function $f(t)$ on \mathbf{P}^1 , we look for a vector field

$$D = f(t)\{g(t)\frac{\partial}{\partial t} + \delta\}$$

such that Y^D is birationally isomorphic to an Enriques surface. In many cases, double fibers of the Enriques surface Y^D exist over the zero points of $g(t)$ by the theory of vector field (cf. Proposition 2.1). In this way, we construct Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$ in Section 5, of type VII in Section 6 and of type VIII in Section 7.

3.2. Enriques surfaces with a quasi-elliptic pencil. By Queen [27, Theorem 2], we have two normal forms for the generic fibers of a quasi-elliptic fiber space over the field $K = k(s)$ with a variable s :

- (1) $u^2 = a + v + cv^2 + dv^4$ with $a, c, d \in K$ and $d \notin K$,
- (2) $u^2 + u = a + dv^4$ with $a, d \in K$ and $d \notin K$.

Here, u, v are variables. Note that the case (3) in Queen [27, Theorem 2] doesn't occur in our case, because the transcendental degree of $K = k(s)$ over k is 1. As for the relative generalized Jacobians of these quasi-elliptic surfaces, Queen [28, Theorem 1] showed the following:

The generalized Jacobian for (1) : $u^2 = v + cv^2 + dv^4$,

The generalized Jacobian for (2) : $u^2 + u = dv^4$.

We use the case (1) to construct our Enriques surfaces. By the change of coordinates $x = 1/v + c$, $y = u/v^2$, the generalized Jacobian for (1) is birationally isomorphic to

$$y^2 = x^3 + c^2x + d,$$

which is a Weierstrass normal form. By Bombieri-Mumford [1], the relative Jacobian of the quasi-elliptic Enriques surface is a rational surface. Therefore, this surface is birationally isomorphic to the rational quasi-elliptic surface in the list of Ito [10, Proposition 5.1].

Starting from Ito's list of rational quasi-elliptic surfaces, we pursue the converse procedure above to construct a candidate of an Enriques surface X , and using the candidate, we construct a vector field D on a rational surface Y such that Y^D is birationally isomorphic to the Enriques surface X . Using this technique, we will construct Enriques surfaces of type \tilde{E}_8 in Section 8, of type $\tilde{E}_7 + \tilde{A}_1$ in Section 9, of type \tilde{D}_8 in Section 10 and of type $\tilde{D}_4 + \tilde{D}_4$ in Section 11.

We will concretely show in the next subsection how to construct a vector field on a rational surface to make an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

3.3. Example: Vector fields for Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$. By Ito [10, Proposition 5.1], we take the rational quasi-elliptic surface defined by

$$y^2 = x^3 + a^4 s^2 x + s^3 \quad \text{with } a \in k.$$

This quasi-elliptic surface has two singular fibers of type I_0^* (namely, of type \tilde{D}_4) over the points on \mathbf{P}^1 defined by $s = 0$ and $s = \infty$. Taking the change of coordinates

$$x = 1/v + a^2 s, \quad y = s^2 u/v^2, \quad s = 1/S$$

we get

$$u^2 = S^4 v + a^2 S^3 v^2 + S v^4$$

Now, we add a term $S^7 + S^3$ and a parameter b ($b \neq 0$) as follows:

$$(3.1) \quad u^2 = b^2 S^4 v + a^2 S^3 v^2 + S v^4 + S^7 + S^3.$$

We need to show that these surfaces are Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$. For this purpose, we take the base change by the Frobenius morphism:

$$S = t^2.$$

Then, the surface becomes

$$u^2 + b^2 t^8 v + a^2 t^6 v^2 + t^2 v^4 + t^{14} + t^6 = 0.$$

Therefore, by this equation we have

$$\{(u + at^3 v + tv^2 + t^7 + t^3)/bt^4\}^2 = v.$$

Now, by the change of coordinates

$$w = (u + at^3 v + tv^2 + t^7 + t^3)/bt^4, \quad v = v, \quad t = t,$$

we have

$$v = w^2.$$

This means we have $k(u, v, t) = k(w, t)$, which is a rational function field of two variables. Since

$$\begin{cases} u = bt^4 w + at^3 w^2 + tw^4 + t^7 + t^3 \\ S = t^2 \\ v = w^2, \end{cases}$$

we have

$$\begin{cases} \frac{\partial u}{\partial w} = bt^4 \\ \frac{\partial u}{\partial t} = at^2 w^2 + w^4 + t^6 + t^2. \end{cases}$$

We put

$$D' = (1/t^3) \left(bt^4 \frac{\partial}{\partial t} + (at^2 w^2 + w^4 + t^6 + t^2) \frac{\partial}{\partial w} \right).$$

Then, we see $D'(u) = 0$, $D'(v) = 0$, $D'(S) = 0$ and $k(t, w)^{D'} = k(u, v, S)$ with the equation (3.1). For the later use, taking new coordinates (x, y) , we consider the change of coordinates

$$x = 1/t, \quad y = t/w.$$

Then, we have

$$\frac{\partial}{\partial t} = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial w} = xy^2 \frac{\partial}{\partial y}.$$

By this change of coordinates, D' becomes

$$(3.2) \quad D = \frac{1}{x^2 y^2} \left(bx^3 y^2 \frac{\partial}{\partial x} + (ax^2 y^2 + x^2 + x^4 y^4 + y^4 + bx^2 y^3) \frac{\partial}{\partial y} \right)$$

where $a, b \in k, b \neq 0$. We will show in Section 11 that the quotient surface with the function field $k(x, y)^D$ is an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

4. EQUATIONS OF ENRIQUES SURFACES AND THEIR AUTOMORPHISMS

4.1. Generalities. Let X be an Enriques surface and assume that X has a structure of a quasi-elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$. Let t be a parameter of an affine line \mathbf{A}^1 in the base curve \mathbf{P}^1 . We denote by C the curve of cusps of the quasi-elliptic fibration, and assume that over the point defined by $t = \infty$ it has a double fiber $2F_\infty$. We assume that

$$(4.1) \quad y^2 = tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t) \quad (g_1(t), g_2(t), g_3(t) \in k[t])$$

is the defining equation of an affine normal surface whose resolution of singularities is isomorphic to the open set $X \setminus (C \cup 2F_\infty)$ of X . Under these conditions, let σ be an automorphism of X which preserves the double fiber $2F_\infty$. Then, for large positive integers m , σ acts on the vector space $L(2mF_\infty)$ associated with the linear system $|2mF_\infty|$. Therefore, σ keeps the structure of the quasi-elliptic fibration $\varphi : X \rightarrow \mathbf{P}^1$, and it acts on the base curve \mathbf{P}^1 with a fixed point at infinity:

$$\begin{array}{ccc} \sigma : \mathbf{P}^1 & \longrightarrow & \mathbf{P}^1 \\ \cup & & \cup \\ \mathbf{A}^1 & & \mathbf{A}^1 \\ t & \mapsto & c_1t + c_2 \end{array}$$

Here, c_1, c_2 are elements of k with $c_1 \neq 0$.

We set $A = k[t, x, y]/(y^2 + tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t))$. Then A is normal by our assumption. As $k[t, x]$ -module, we have

$$(4.2) \quad A = k[t, x] \oplus k[t, x]y,$$

which is a free $k[t, x]$ -module. Since σ preserve C and $2F_\infty$, σ acts on the open set $X \setminus (C \cup 2F_\infty)$ of X .

Lemma 4.1. σ induces an automorphism of $\text{Spec}(A)$.

Proof. We consider the change of coordinates

$$u = \frac{1}{x}, \quad v = \frac{y}{x^2}.$$

Then, the equation becomes $v^2 = t + g_1(t)u^2 + g_2(t)u^3 + g_3(t)u^4$, and the curve C of cusps is given by $u = 0$. On the curve C , the affine surface is nonsingular. Therefore, the open set $X \setminus (C \cup 2F_\infty)$ is constructed by some blowing-ups of $\text{Spec}(A)$:

$$\pi : X \setminus (C \cup 2F_\infty) \rightarrow \text{Spec}(A).$$

Note that π is surjective. Since σ is an automorphism of $X \setminus (C \cup 2F_\infty)$, we have a morphism

$$(\pi, \pi \circ \sigma) : X \setminus (C \cup 2F_\infty) \rightarrow \text{Spec}(A) \times \text{Spec}(A).$$

We denote by Γ the image of the morphism $(\pi, \pi \circ \sigma)$. We denote by p_1 (resp. p_2) the first projection (resp. the second projection) : $p_i : \text{Spec}(A) \times \text{Spec}(A) \longrightarrow \text{Spec}(A)$ ($i = 1, 2$). Then, restricting the projection p_1 to Γ , we have a morphism

$$p_1|_{\Gamma} : \Gamma \longrightarrow \text{Spec}(A).$$

Since $\text{Spec}(A)$ is affine, the exceptional curves by blowing-ups collapse by the morphism $(\pi, \pi \circ \sigma)$. Therefore, the morphism $p_1|_{\Gamma}$ is a finite birational morphism. Since $\text{Spec}(A)$ is normal by our assumption, we see that by the Zariski main theorem $p_1|_{\Gamma}$ is an isomorphism. Therefore, we have a morphism $p_2|_{\Gamma} \circ p_1|_{\Gamma}^{-1} : \text{Spec}(A) \longrightarrow \text{Spec}(A)$ which is the induced automorphism by σ . \square

By this lemma, σ acts on $\text{Spec}(A)$ and induces an automorphism

$$(4.3) \quad \sigma^* : A \longrightarrow A.$$

Now we consider the generic fiber of $\varphi : X \longrightarrow \mathbf{P}^1$. It is a curve of genus one over $k(t)$ whose affine part is given by the equation (4.1). The curve C of cusps gives a point P_{∞} of degree 2 on the curve of genus one. We denote by $\tilde{L}(P_{\infty})$ the vector space over $k(t)$ associated with the linear system $|P_{\infty}|$ on the curve of genus one. By the Riemann-Roch theorem, we have $\dim \tilde{L}(P_{\infty}) = 2$ and we see that 1 and x give the basis of $\tilde{L}(P_{\infty})$. Since σ preserves the curve C of cusps, $\sigma^*(x)$ is contained in $\tilde{L}(P_{\infty})$. Therefore, there exist $d_1(t), d_2(t) \in k(t)$ such that

$$\sigma^*(x) = d_1(t)x + d_2(t).$$

By (4.2) and (4.3), there exist $d_3(t, x), d_4(t, x) \in k[t, x]$ such that

$$\sigma^*(x) = d_3(t, x) + d_4(t, x)y.$$

Therefore, considering $\sigma^*(x)^2$, we have

$$d_1(t)^2 x^2 + d_2(t)^2 = d_3(t, x)^2 + d_4(t, x)^2 (tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t)).$$

Since the right-hand-side is in $k[t, x]$, we see that $d_1(t)$ and $d_2(t)$ are also polynomials of t . Therefore, we see that σ is of the following form:

$$(4.4) \quad \sigma : \begin{cases} t & \mapsto c_1 t + c_2 \quad (c_1, c_2 \in k; c_1 \neq 0) \\ x & \mapsto d_1(t)x + d_2(t) \quad (d_1(t), d_2(t) \in k[t]; d_1(t) \neq 0) \\ y & \mapsto e_1(t, x)y + e_2(t, x) \quad (e_1(t, x), e_2(t, x) \in k[t, x]; e_1(t, x) \neq 0) \end{cases}$$

Remark 4.2. Let X be an Enriques surface which has a structure of elliptic or quasi-elliptic fibration $\varphi : X \longrightarrow \mathbf{P}^1$ defined by

$$y^2 + g_0(t)y = tx^4 + g_1(t)x^2 + g_2(t)x + g_3(t)$$

with $g_0(t), g_1(t), g_2(t), g_3(t) \in k[t]$. Here, t is a parameter of an affine line \mathbf{A}^1 in the base curve \mathbf{P}^1 . We denote by C the 2-section defined by $x = \infty$, and by F_{∞} the fiber over the point on \mathbf{P}^1 defined by $t = \infty$. We assume that the equation is the defining equation

of an affine normal surface whose resolution of singularities is isomorphic to the open set $X \setminus (C \cup F_\infty)$ of X . Under these conditions, let σ be an automorphism of X which preserves the curve C and the fiber F_∞ . Then, the automorphism σ is also expressed as the form (4.4), and a similar argument to the above works.

We use the following trivial lemma.

Lemma 4.3. *$k[x, y]$ is a free $k[x^2, y^2]$ -module of rank 4. A basis is given by $1, x, y, xy$.*

4.2. List of equations and automorphisms. In this subsection, we list up the equations of Enriques surfaces X with finite automorphism group and their automorphism groups. We will use these equations to calculate the automorphism group in cases of type $\tilde{E}_6 + \tilde{A}_2$ (supersingular), type \tilde{E}_8 (supersingular and classical), type $\tilde{E}_7 + \tilde{A}_1$ (supersingular), type \tilde{D}_8 (supersingular and classical) and type $\tilde{D}_4 + \tilde{D}_4$. We will give the proof of this list in Examples 4.3, 4.4, 4.5 and in Theorem 5.9, Theorem 8.4, Theorem 8.9, Theorem 9.10, Theorem 10.5, Theorem 10.11, Theorem 11.4. For the remaining cases, we do not use this list to determine the automorphism groups and hence omit the details.

(1) Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$.

(i) Supersingular case:

$$y^2 + ty = tx^4 + x^3 + t^3x + t^7, \quad \text{Aut}(X) \cong \langle \sigma, \tau, \rho \rangle \cong \mathbf{Z}/5\mathbf{Z} \times \mathfrak{S}_3,$$

$$\text{where } \sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \zeta^4 x \\ y \mapsto \zeta y, \end{cases} \quad \tau : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + t, \end{cases} \quad \text{and } \rho \text{ is an automorphism induced from the}$$

action of a section of order 3 of the relative Jacobian of the elliptic fibration on X with singular fibers (IV, IV*). Here, ζ is a primitive fifth root of unity and $\langle \tau, \rho \rangle \cong \mathfrak{S}_3$.

(ii) Classical case:

$$y^2 + c^2txy + \beta c^3t^2y = tx^4 + c^2t^3x^2 + (c^3t^4 + c^5\alpha t^3)x + t^7 + t^3 = 0,$$

where $c = \frac{1}{a + \sqrt[4]{a^3}}$ ($a \neq 0, 1$), α is a root of $z^8 + z^6 + z^5 + a^2z^4 + a^4z^3 + a^8z^2 + a^{16} = 0$,
and $\beta = \frac{\alpha^2 + a^4}{\alpha}$.

$$\text{Aut}(X) \cong \langle \sigma, \tau \rangle \cong \mathfrak{S}_3, \text{ where } \sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + c^2tx + \beta c^3t^2 \end{cases} \quad \text{and } \tau \text{ is an automor-}$$

phism induced from the action of a section of order 3 of the relative Jacobian of the elliptic fibration on X with singular fibers (IV, IV*).

(2) Enriques surfaces of type VII:

$$\begin{aligned} y^2 = & t(t+1)(t+a^2)(t+b^2)xy + \{(ab+1)t+ab\}(t+1)(t+a^2)(t+b^2)y + tx^4 + \\ & \{(ab+1)t+ab\}(t+1)(t+a^2)(t+b^2)x^3 + \{t^2+(t+1)(t+a^2)(t+b^2)\}(t+1)(t+a^2)(t+b^2)x^2 \\ & + \{(ab+1)t+ab\}t(t+1)(t+a^2)(t+b^2)x + t^3 + t^3(t+1)(t+a^2)(t+b^2) \\ & + t(t+1)^2(t+a^2)^2(t+b^2)^2 + t(t+1)^3(t+a^2)^3(t+b^2)^3, \end{aligned}$$

where $a, b \in k$, $a + b = ab$, $a^3 \neq 1$. In this case we calculated $\text{Aut}(X)$ from the dual graph in [13].

(3) Enriques surfaces of type VIII:

$$y^2 = tx^4 + at^2x^3 + at^3(t+1)^2x + t^7 + t^3 \quad (a \neq 0).$$

In this case we calculate $\text{Aut}(X)$ from the dual graph in Section 7.

(4) Enriques surfaces of type \tilde{E}_8 .

(i) Supersingular case:

$$y^2 = tx^4 + x + t^7, \quad \text{Aut}(X) \cong \langle \sigma \rangle \cong \mathbf{Z}/11\mathbf{Z}$$

where $\sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \zeta^7 x \\ y \mapsto \zeta^9 y \end{cases}$ and ζ is a primitive 11-th root of unity.

(ii) Classical case:

$$y^2 = tx^4 + at^3x + t^7 + t^3 \quad (a \neq 0), \quad \text{Aut}(X) \simeq \{1\}.$$

(5) Enriques surfaces of type $\tilde{E}_7 + \tilde{A}_1$

(i) Supersingular case:

$$\begin{aligned} y^2 + y &= tx^4 + ax + t^7 \quad (a \neq 0), \\ \text{Aut}(X) &\cong \langle \sigma \rangle \cong \mathbf{Z}/2\mathbf{Z} \text{ if } a^7 \neq 0, 1, \\ \text{Aut}(X) &\cong \langle \sigma \rangle \cong \mathbf{Z}/14\mathbf{Z} \text{ if } a^7 = 1. \end{aligned}$$

By the change of coordinates $t \mapsto t + a^4$, $y \mapsto y + a^2x^2 + ax$, $x \mapsto x$, the equation becomes

$$y^2 + y = tx^4 + (t + a^4)^7$$

and σ is given by

$$\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + 1. \end{cases} \quad \text{if } a^7 \neq 0, 1,$$

$$\sigma : \begin{cases} t \mapsto \zeta t \\ x \mapsto \frac{1}{\sqrt[4]{\zeta}}x + \frac{(\sqrt[4]{\zeta}+1)}{\sqrt[4]{\zeta}}a^6 + \frac{(\sqrt[4]{\zeta^5}+1)}{\sqrt[4]{\zeta}}a^2t \\ y \mapsto y + 1 + (1 + \zeta^2)a^6t^2 + (1 + \zeta^3)a^2t^3 \end{cases} \quad \text{if } a^7 = 1,$$

where ζ is a primitive 7-th root of unity.

(ii) Classical one with singular fibers of type $(2\text{III}^*, \text{III})$:

$$y^2 + at^2y = tx^4 + bt^3x + t^7 + t^3 \quad (a \neq 0, b \neq 0), \quad \text{Aut}(X) \cong \langle \sigma \rangle \cong \mathbf{Z}/2\mathbf{Z}$$

where $\sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + at^2. \end{cases}$

(iii) Classical one with singular fibers of type $(2\text{III}^*, 2\text{III})$:

$$y^2 + at^2y = tx^4 + t^7 + t^3 \quad (a \neq 0), \quad \text{Aut}(X) \cong \langle \sigma \rangle \cong \mathbf{Z}/2\mathbf{Z}$$

$$\text{where } \sigma : \begin{cases} t \mapsto t \\ x \mapsto x \\ y \mapsto y + at^2. \end{cases}$$

(6) Enriques surfaces of type \tilde{D}_8

(i) Supersingular case:

$$y^2 = tx^4 + tx^2 + ax + t^7 \quad (a \neq 0), \quad \text{Aut}(X) \cong \langle \{\sigma_{\omega, \alpha}\} \rangle \cong Q_4$$

$$\sigma_{\omega, \alpha} : \begin{cases} t \mapsto t + \omega \\ x \mapsto x + \alpha + \omega^2 t \\ y \mapsto y + \omega^2 x^2 + \omega^2 x + \omega^2 t^3 + \sqrt{a\alpha} + \sqrt{a} \end{cases}$$

Here, ω is a primitive cube root of unity and α is a root of the equation $z^2 + z + \omega\sqrt{a} + 1 = 0$.

(ii) Classical case:

$$y^2 = tx^4 + at^3x^2 + bt^3x + t^7 + t^3 \quad (a \neq 0, b \neq 0), \quad \text{Aut}(X) \cong \langle \sigma \rangle \cong \mathbf{Z}/2\mathbf{Z}$$

$$\text{where } \sigma : \begin{cases} t \mapsto t \\ x \mapsto x + \sqrt{at} \\ y \mapsto y + \sqrt[4]{a}\sqrt{b}t^2. \end{cases}$$

(7) Classical Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$

$$y^2 = tx^4 + at^3x^2 + bt^4x + t^7 + t^3 \quad (b \neq 0), \quad \text{Aut}(X) \cong \langle \{\sigma_\alpha\}, \tau \rangle \cong (\mathbf{Z}/2\mathbf{Z})^3,$$

$$\text{where } \sigma_\alpha : \begin{cases} t \mapsto t \\ x \mapsto x + \alpha t \\ y \mapsto y, \end{cases} \quad \tau : \begin{cases} t \mapsto 1/t \\ x \mapsto x/t^2 \\ y \mapsto y/t^5, \end{cases}$$

and α is a root of the equation $z^3 + az + b = 0$.

4.3. Example 1. We calculate the defining equation of classical Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$. As in (5.3), (5.5) in Section 5, we take the elliptic surface defined by $y^2 + xy + t^2y = x^3$, and a vector field $D = (t+a)\frac{\partial}{\partial t} + (x+t^2)\frac{\partial}{\partial x}$ on it. Set $T = t^2$, $u = (t+a)x + t^3$, and $v = (t+a)^3(y+x^2)$. Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ and $k(t, x, y)^D = k(T, u, v)$. We have a relation

$$\begin{aligned} v^2 + (T+a^2)uv + a(T+a^2)Tv \\ = (T+a^2)u^4 + (T^2+a^4)(Tu^2+T^4) + T^6(T+a^2), \end{aligned}$$

and this equation defines our classical Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$. We put $c = 1/(a + \sqrt[4]{a^3})$, and consider the change of new coordinates

$$\begin{cases} T = \frac{1}{c^2}t + a^2 \\ u = \frac{1}{c^3}x + \frac{\beta+a}{c^2}t + a^3 \\ v = \frac{1}{c^7}y + \frac{\delta}{c^5}tx + \frac{1}{c^6}t^3 + \frac{\alpha+\beta\delta}{c^4}t^2. \end{cases}$$

Here, δ is a root of the equation $z^2 + z + a^2 = 0$, α is a root of the equation $z^8 + z^6 + z^5 + a^2 z^4 + a^4 z^3 + a^8 z^2 + a^{16} = 0$, and $\beta = \frac{\alpha^2 + a^4}{\alpha}$. Then, we get the normal form

$$y^2 + c^2 txy + \beta c^3 t^2 y = tx^4 + c^2 t^3 x^2 + (c^3 t^4 + c^5 \alpha t^3)x + t^7 + t^3 = 0.$$

4.4. Example 2. We calculate the defining equation of classical Enriques surfaces of type VII. In [13], to construct Enriques surfaces of type VII, we use an elliptic surface defined by $y^2 + t^2 xy + y = x^3 + x^2 + t^2$ and a vector field $D = (t + a)(t + b)\frac{\partial}{\partial t} + \frac{t^2 x + 1}{t + 1}\frac{\partial}{\partial x}$, $a, b \in k$, $a + b = ab$, $a^3 \neq 1$. Put

$$X = (t + 1)(t + a)(t + b)x + t, \quad Y = (t + 1)(t + a)(t + b) + tx^2, \quad T = t^2.$$

Then $k(x, y, t)^D = k(X, Y, T)$. Thus, replacing X , $Y(T + 1)(T + a^2)(T + b^2)$, T by x , y , t , respectively, we have the equation of Enriques surfaces of type VII.

4.5. Example 3. We calculate the defining equation of classical Enriques surfaces of type VIII. We consider the elliptic surface Y defined by

$$y^2 + txy + ty = x^3 + x^2$$

(see Section 7, (7.1)). Then, we have

$$\frac{\partial y}{\partial t} = \frac{y}{t}, \quad \frac{\partial y}{\partial x} = \frac{ty + x^2}{t(x + 1)}.$$

Therefore, considering x, y as local parameters instead of x, t , and using $t = \frac{x^3 + x^2 + y^2}{(x + 1)y}$, we have

$$\begin{aligned} D &= t(at + 1)\frac{\partial}{\partial t} + (x + 1)\frac{\partial}{\partial x} \\ &= t(at + 1)\left\{\frac{(x + 1)y^2}{x^3 + x^2 + y^2}\frac{\partial}{\partial y}\right\} + (x + 1)\left\{\frac{\partial}{\partial x} + \frac{ty + x^2}{t(x + 1)}\frac{\partial}{\partial y}\right\} \\ &= \frac{x^3 + x^2 + y^2}{(x + 1)y}\left\{a\frac{x^3 + x^2 + y^2}{(x + 1)y} + 1\right\}\left\{\frac{(x + 1)y^2}{x^3 + x^2 + y^2}\frac{\partial}{\partial y}\right\} \\ &\quad + (x + 1)\frac{\partial}{\partial x} + (x + 1)\frac{(x + 1)y}{x^3 + x^2 + y^2}\frac{1}{x + 1}\left\{\frac{x^3 + x^2 + y^2}{(x + 1)y}y + x^2\right\}\frac{\partial}{\partial y} \\ &= \frac{1}{(x + 1)(x^3 + x^2 + y^2)}\left\{a(x^6 + x^4 + y^4) + x^4 y + x^2 y\right\}\frac{\partial}{\partial y} \\ &\quad + (x^5 + x^4 + x^2 y^2 + x^3 + x^2 + y^2)\frac{\partial}{\partial x} \end{aligned}$$

with $a \neq 0$. Putting

$$T = x^2, X = y^2, z = ax^7 + ax^5 + ay^4x + x^5y + x^3y + x^4y + x^2y^3 + x^2y + y^3,$$

we have $D(T) = D(X) = D(z) = 0$, and we have an equation

$$z^2 = a^2 T X^4 + (T^2 + 1)X^3 + (T^5 + T^4 + T^3 + T^2)X + a^2 T^5 + a^2 T^7,$$

which gives birationally the equation of Y^D . We consider the change of coordinates defined by

$$y = \frac{z}{a} + X^2 + T^3 + T^2, \quad t = T + 1, \quad x = X.$$

with new variables x, y, t . Then the equation becomes

$$y^2 = tx^4 + \frac{1}{a^2}t^2x^3 + \frac{1}{a^2}t^3(t+1)^2x + t^7 + t^3$$

For the sake of simplicity, we replace $\frac{1}{a^2}$ by a . Then we have the normal form

$$y^2 = tx^4 + at^2x^3 + at^3(t+1)^2x + t^7 + t^3.$$

Remark 4.4. This surface has an involution defined by

$$t \mapsto \frac{1}{t}, \quad x \mapsto \frac{x}{t^2}, \quad y \mapsto \frac{y}{t^5}.$$

Other results on the defining equations and their groups of automorphisms in Subsection 4.2 are obtained in a similar way.

5. ENRIQUES SURFACES OF TYPE $\tilde{E}_6 + \tilde{A}_2$

From Section 5 to Section 11, we will construct the examples of Enriques surfaces given in Theorem 1.1 (B) and Theorem 1.2 (B). First, we consider the cases where the Enriques surfaces have a special elliptic fibration with a desired double fiber, that is, the cases of type $\tilde{E}_6 + \tilde{A}_2$, of type VII and of type VIII. Next we consider the remaining cases where the Enriques surfaces have a special quasi-elliptic fibration with a desired double fiber. In this section, we give Enriques surfaces of type $\tilde{E}_6 + \tilde{A}_2$.

5.1. Supersingular case. We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \rightarrow \mathbf{P}^1$ associated with a Weierstrass equation

$$y^2 + sy = x^3$$

with a parameter s . This surface is a unique rational elliptic surface with a singular fiber of type IV over the point given by $s = 0$ and a singular fiber of type IV* over the point given by $s = \infty$ (Lang [18, §2]). Note that all nonsingular fibers are supersingular elliptic curves. We consider the base change of $\psi : R \rightarrow \mathbf{P}^1$ by $s = t^2$. Then, we have the elliptic surface defined by

$$(5.1) \quad y^2 + t^2y = x^3.$$

We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$(5.2) \quad f : \tilde{R} \rightarrow \mathbf{P}^1.$$

By considering the change of coordinates defined by $x' = x/t^2, y' = y/t^3, t' = 1/t$, we have

$$y'^2 + t'y' = x'^3.$$

Thus the surface \tilde{R} is isomorphic to R . The rational elliptic surface $f : \tilde{R} \rightarrow \mathbf{P}^1$ has a singular fiber of type IV* over the point given by $t = 0$ and a singular fiber of type IV over the point given by $t = \infty$.

The elliptic surface $f : \tilde{R} \longrightarrow \mathbf{P}^1$ has three sections s_i ($i = 0, 1, 2$) given as follows:

s_0 : the zero section.

s_1 : $x = y = 0$.

s_2 : $x = 0, y = t^2$.

On the singular elliptic surface (5.1), we denote by F_0 the fiber over the point defined by $t = 0$, and by F_∞ the fiber over the point defined by $t = \infty$. Both F_0 and F_∞ are irreducible, and on each F_i ($i = 0, \infty$) the surface (5.1) has only one singular point P_i . The surface \tilde{R} is the surface obtained by the minimal resolution of singularities of the surface (5.1). We denote the proper transform of F_i on \tilde{R} again by F_i , if confusion doesn't occur. We have six exceptional curves $E_{0,k}$ ($k = 1, 2, \dots, 6$) over the point P_0 such that F_0 and these six exceptional curves make a singular fiber of type IV^* of the elliptic surface $f : \tilde{R} \longrightarrow \mathbf{P}^1$ as follows: The blowing-up at the singular point P_0 gives one exceptional curve $E_{0,1}$, and the surface is nonsingular along F_0 and has a unique singular point $P_{0,1}$ on $E_{0,1}$. The blowing-up at the singular point $P_{0,1}$ gives two exceptional curves $E_{0,2}$ and $E_{0,3}$. We denote the proper transform of $E_{0,1}$ by $E_{0,1}$. The three curves $E_{0,1}$, $E_{0,2}$ and $E_{0,3}$ meet at one point $P_{0,2}$ which is a singular point of the obtained surface. The blowing-up at the singular point $P_{0,2}$ again gives two exceptional curves $E_{0,4}$ and $E_{0,5}$. The three curves $E_{0,1}$, $E_{0,4}$ and $E_{0,5}$ meet at one point $P_{0,3}$ which is a singular point of the obtained surface. The curve $E_{0,2}$ (resp. $E_{0,3}$) intersects $E_{0,4}$ (resp. $E_{0,5}$) and does not meet other curves. Finally, the blowing-up at the singular point $P_{0,3}$ gives an exceptional curve $E_{0,6}$ and the obtained surface is nonsingular over these curves. The curve $E_{0,6}$ meets $E_{0,1}$, $E_{0,4}$ and $E_{0,5}$ transversally. The dual graph of the curves $F_0, E_{0,1}, \dots, E_{0,6}$ is of type \tilde{E}_6 . The cycle

$$F_0 + E_{0,2} + E_{0,3} + 2(E_{0,1} + E_{0,4} + E_{0,5}) + 3E_{0,6}$$

forms a singular fiber of type IV^* . On the other hand, the blowing-up at the singular point P_∞ gives two exceptional curves $E_{\infty,1}$ and $E_{\infty,2}$. The obtained surface is now nonsingular, that is, nothing but \tilde{R} . The three curves F_∞ , $E_{\infty,1}$ and $E_{\infty,2}$ form a singular fiber of type IV . The configuration of these curves is as in the following Figure 1.

The sections s_i has the self-intersection number -1 and others have the self-intersection number -2 .

Now, we consider a rational vector field on \tilde{R} induced from

$$D = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}.$$

Then, we have $D^2 = 0$, that is, D is 2-closed. However D has an isolated singularity at the point P which is the singular point of the fiber of type IV , that is, the intersection point of three curves F_∞ , $E_{\infty,1}$ and $E_{\infty,2}$ (note that (t, x) is not a local parameter along the fiber defined by $t = 0$). To resolve this singularity, we first blow up at P . Denote by $E_{\infty,3}$ the exceptional curve. We denote the proper transforms of F_∞ , $E_{\infty,1}$ and $E_{\infty,2}$ by the same

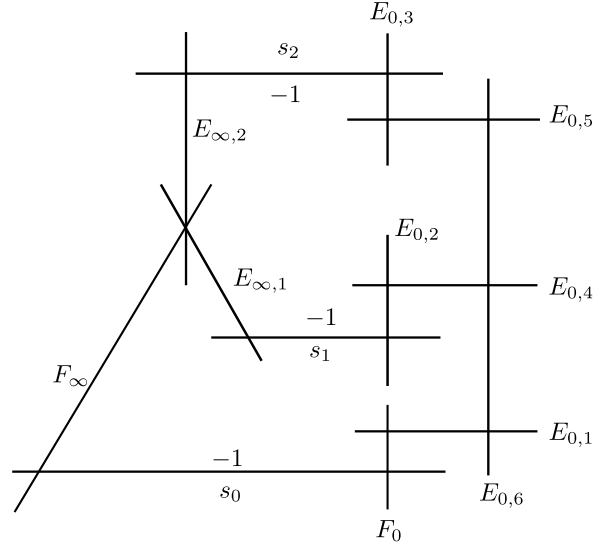


FIGURE 1

symbols. Then blow up at three points $E_{\infty,3} \cap (F_{\infty} + E_{\infty,1} + E_{\infty,2})$. Let Y be the obtained surface and $\psi : Y \rightarrow \tilde{R}$ the successive blowing-ups. We denote by $E_{\infty,4}$, $E_{\infty,5}$ or $E_{\infty,6}$ the exceptional curve over the point $E_{\infty,3} \cap F_{\infty}$, $E_{\infty,3} \cap E_{\infty,1}$ or $E_{\infty,3} \cap E_{\infty,2}$ respectively. Then we have the following Figure 2. In this Figure 2 we give the self-intersection numbers of the curves except for the curves with the self-intersection number -2 , and the thick lines are integral curves with respect to D .

Now, according to the above blowing-ups, we see the following:

Lemma 5.1. (i) *The divisorial part (D) on Y is given by*

$$-2(E_{0,1} + E_{0,4} + E_{0,5} + E_{0,6} + E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_{\infty} + E_{\infty,1} + E_{\infty,2}).$$

(ii) *The integral curves in Figure 2 are*

$$E_{0,1}, E_{0,4}, E_{0,5}, F_{\infty}, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}.$$

Lemma 5.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by*

$$K_Y = -2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_{\infty} + E_{\infty,1} + E_{\infty,2}).$$

(iii) $K_Y \cdot (D) = -4$.

Lemma 5.3. *D is divisorial and the quotient surface Y^D is nonsingular.*

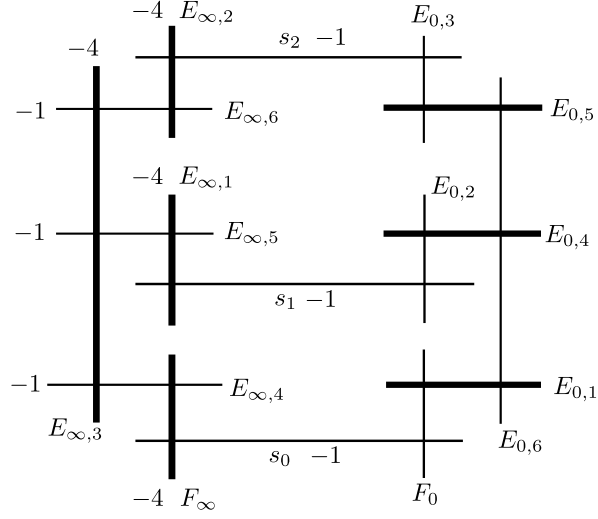


FIGURE 2

Proof. Since \tilde{R} is a rational elliptic surface and Y is the blowing-ups at 4 points, we have $c_2(Y) = 16$. Using $(D)^2 = -12$, $K_Y \cdot (D) = -4$ and the equation (2.2), we have

$$16 = c_2(Y) = \deg\langle D \rangle - K_Y \cdot (D) - (D)^2 = \deg\langle D \rangle + 4 + 12.$$

Therefore, we have $\deg\langle D \rangle = 0$. This means that D is divisorial, and that Y^D is nonsingular. \square

Let $\pi : Y \rightarrow Y^D$ be the natural map. By the result on the canonical divisor formula (2.1), we have

$$K_Y = \pi^* K_{Y^D} + (D).$$

Lemma 5.4. (i) *The images of the curves $E_{0,1}, E_{0,4}, E_{0,5}$ in Y^D are exceptional curves.*
(ii) *The self-intersection numbers of the images of $F_0, E_{0,2}, E_{0,3}, E_{0,6}$ in Y^D are -4 .*
(iii) *The self-intersection numbers of the images of $F_{\infty}, E_{\infty,i}$ ($i = 1, \dots, 6$) and three sections s_i ($i = 0, 1, 2$) in Y^D are -2 .*

Proof. The assertions follows from Proposition 2.1 and Lemma 5.1, (ii). \square

Let $E'_{0,1}, E'_{0,4}, E'_{0,5}, E'_{0,6}$ be the image of $E_{0,1}, E_{0,4}, E_{0,5}, E_{0,6}$ in Y^D , respectively. Then we have the following Figure 3 in which we give the self-intersection numbers of the curves except the curves with the self-intersection number -2 .

Let

$$\varphi_1 : Y^D \rightarrow X'$$

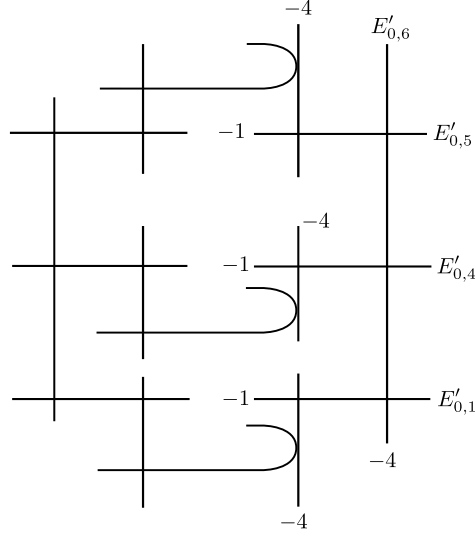


FIGURE 3

be the blowing-downs of $E'_{0,1}, E'_{0,4}, E'_{0,5}$. Then the image of $E'_{0,6}$ in X' is an exceptional curve. Let

$$\varphi_2 : X' \rightarrow X$$

be the blowing-down of this exceptional curve. Now we have the following diagram

$$\begin{array}{ccc} Y^D & \xleftarrow{\pi} & Y \\ \varphi_1 \downarrow & & \downarrow \psi \\ X' & & \tilde{R} \\ \varphi_2 \downarrow & & \\ X & & \end{array}$$

We have thirteen (-2) -curves E_1, \dots, E_{13} with the self-intersection number -2 which form the following Figure 4.

Then, we have

$$\begin{aligned} K_{Y^D} &= \varphi_1^*(K_{X'}) + E_{0,1} + E_{0,4} + E_{0,5} \\ &= \varphi_1^* \circ \varphi_2^*(K_X) + E_{0,6} + 2(E_{0,1} + E_{0,4} + E_{0,5}). \end{aligned}$$

Lemma 5.5. *The canonical divisor K_X of X is numerically equivalent to 0.*

Proof. By Lemma 5.2, (ii),

$$K_Y = -2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}) - (F_{\infty} + E_{\infty,1} + E_{\infty,2}).$$

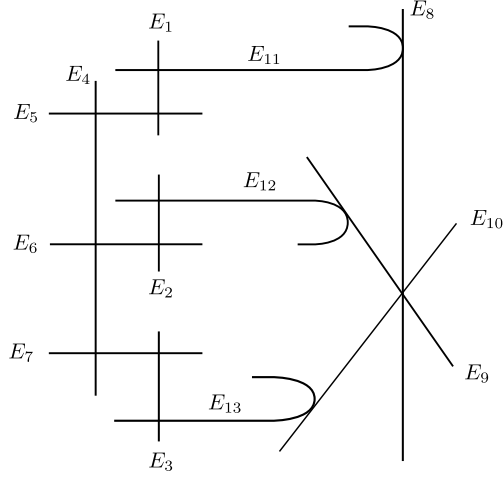


FIGURE 4

On the other hand,

$$\begin{aligned} K_Y &= \pi^* K_{Y^D} + (D) = \pi^*(\varphi_1^* \circ \varphi_2^*(K_X) + E_{0,6} + 2(E_{0,1} + E_{0,4} + E_{0,5})) + (D) = \\ &= \pi^*(\varphi_1^* \circ \varphi_2^*(K_X)) + 2(E_{0,6} + E_{0,1} + E_{0,4} + E_{0,5}) + (D) = \pi^*(\varphi_1^* \circ \varphi_2^*(K_X)) + K_Y. \end{aligned}$$

Here we use the fact that $E_{0,1}, E_{0,4}, E_{0,5}$ are integral and $E_{0,6}$ is non-integral (Lemma 5.1, (ii) and Lemma 2.1). Therefore, K_X is numerically equivalent to zero. \square

Lemma 5.6. *The surface X has $b_2(X) = 10$.*

Proof. Since $\pi : Y \rightarrow Y^D$ is finite and purely inseparable, the étale cohomology of \tilde{Y} is isomorphic to the étale cohomology of Y^D . Therefore, we have $b_1(Y^D) = b_1(Y) = 0$, $b_3(Y^D) = b_3(Y) = 0$ and $b_2(Y^D) = b_2(Y) = 14$. Since $\varphi_2 \circ \varphi_1$ is the blowing-downs of four exceptional curves, we see $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$ and $b_2(X) = 10$. \square

Theorem 5.7. *With the notation above, X is a supersingular Enriques surface.*

Proof. Since K_X is numerically trivial, X_a is minimal and the Kodaira dimension $\kappa(X)$ is equal to 0. Since $b_2(X) = 10$, X is an Enriques surface. Since \tilde{Y} is a rational surface, X_a is either supersingular or classical. Consider the elliptic fibration $g : X \rightarrow \mathbf{P}^1$ induced by $f : \tilde{R} \rightarrow \mathbf{P}^1$. Note that the fiber over the point given by $t = \infty$ is a double fiber of type IV^* and the fiber over the point given by $t = 0$ is simple. Since the other fibers are smooth and supersingular elliptic curves by Lemma 2.6, they are simple by Proposition 2.5. Therefore X is a supersingular Enriques surface by Proposition 2.5. \square

The dual graph of the thirteen (-2) -curves E_1, \dots, E_{13} is as in the following Figure 5.

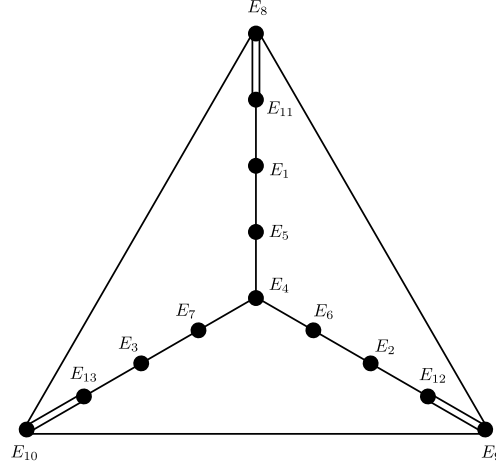


FIGURE 5

We now have the following theorem.

Theorem 5.8. *The Enriques surface X contains exactly thirteen (-2) -curves.*

Proof. Consider the dual graph Γ of 13 (-2) -curves in Figure 5. We can easily prove that any maximal parabolic subdiagram in Γ is of type $\tilde{E}_6 \oplus \tilde{A}_2$ or of type $\tilde{E}_7 \oplus \tilde{A}_1$. It follows from Propositions 2.2, 2.12 and 2.13 that $\text{Aut}(X)$ is finite and X contains exactly 13 (-2) -curves. \square

On X_a , there exist exactly one elliptic fibration with singular fibers of type $(2\text{IV}^*, \text{IV})$ defined by the linear system $|E_8 + E_9 + E_{10}|$ and three quasi-elliptic fibrations with singular fibers of type $(\text{III}^*, 2\text{III})$ defined by $|2(E_9 + E_{12})|$, $|2(E_8 + E_{11})|$, $|2(E_{10} + E_{13})|$ respectively.

Theorem 5.9. *The automorphism group $\text{Aut}(X)$ is isomorphic to $\mathbf{Z}/5\mathbf{Z} \times \mathfrak{S}_3$ and the numerically trivial automorphism group $\text{Aut}_{nt}(X)$ is isomorphic to $\mathbf{Z}/5\mathbf{Z}$.*

Proof. To calculate $\text{Aut}(X)$ we first give an equation of X mentioned in Section 4 and then determine its automorphism group. As in Subsection 4.1, we consider the elliptic surface defined by $y^2 + t^2y = x^3$ and the vector field $D = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}$. Put $T = t^2$, $u = x + t^3$, $v = y + tx^2$. Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ and we have the relation $v^2 + Tv = Tu^4 + u^3 + T^3u + T^7$. Since we have $k(x, y, t)^D = k(u, v, T)$, the quotient surface by D is birationally isomorphic to the surface defined by $v^2 + Tv =$

$Tu^4 + u^3 + T^3u + T^7$. We replace variables u, v, T by new variables x, y, t , respectively for convenience. Then, the equation $y^2 + ty = tx^4 + x^3 + t^3x + t^7$ gives a normal affine surface. Note that the minimal normal completion of this surface is a normal elliptic surface $f : X \rightarrow \mathbf{P}^1$ which is birationally isomorphic to our Enriques surface. Set

$$A = k[t, x, y]/(y^2 + ty + tx^4 + x^3 + t^3x + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type IV* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2) -curves, σ preserves $2F_\infty$. Therefore, σ preserves the structure of this elliptic surface. Since there are three 2-sections for this elliptic surface by the configuration of (-2) -curves, σ acts on these three 2-sections as a permutation. We denote by \tilde{C} be the 2-section at infinity and assume that σ preserves \tilde{C} . Then, as in the case of a quasi-elliptic surface, σ has the form in (4.4) in Subsection 4.1. Moreover, this elliptic surface has a singular fiber over the point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 = ty + tx^4 + x^3 + t^3x + t^7$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(ty + tx^4 + x^3 + t^3x + t^7) + e_2(t, x)^2 + c_1t(e_1(t, x)y + e_2(t, x)) \\ &= c_1t(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t))^3 \\ & \quad + (c_1t)^3(d_1(t)x + d_2(t)) + (c_1t)^7. \end{aligned}$$

A is a free $k[t, x]$ -module, and 1 and y are linearly independent over $k[t, x]$. Taking the coefficient of y , we have $e_1(t, x)^2t + c_1te_1(t, x) = 0$. Since $e_1(t, x) \neq 0$, we have $e_1(t, x) = c_1$, which is a constant. Therefore, we have

$$\begin{aligned} & c_1^2(tx^4 + x^3 + t^3x + t^7) + e_2(t, x)^2 + c_1te_2(t, x) \\ &= c_1t(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t))^3 \\ & \quad + (c_1t)^3(d_1(t)x + d_2(t)) + (c_1t)^7. \end{aligned}$$

As a polynomial of x , if $e_2(t, x)$ has a term of degree greater than or equal to 3, then $e_2(t, x)^2$ has a term greater than or equal to 6. We cannot kill this term in the equation. Therefore, we can put $e_2(t, x) = a_0(t) + a_1(t)x + a_2(t)x^2$ with $a_0(t), a_1(t), a_2(t) \in k[t]$. We take terms which contain only the variable t . Then, we have an equality

$$c_1^2t^7 + a_0(t)^2 + c_1ta_0(t) = c_1td_2(t)^4 + d_2(t)^3 + c_1^3t^3d_2(t) + c_1^7t^7.$$

Put $\deg d_2(t) = \ell$. Suppose $\ell \geq 2$. Then, the right-hand-side has an odd term whose degree is equal to $4\ell + 1 \geq 9$. Therefore, the left-hand-side must have an odd term which is of degree $4\ell + 1$. This means $\deg a_0(t) = 4\ell + 1$. However, in the equation we cannot kill the term of degree $8\ell + 2$ which comes from $a_0(t)^2$. Therefore, we can put $d_2(t) = b_0 + b_1t$ with $b_0, b_1 \in k$. Then, the equation becomes

$$\begin{aligned} & a_0(t)^2 + c_1ta_0(t) + c_1^2t^7 \\ &= c_1b_0^4t + c_1b_1^4t^5 + b_0^3 + b_0^2b_1t + b_0b_1^2t^2 + b_1^3t^3 + c_1^3b_0t^3 + c_1^3b_1t^4 + c_1^7t^7. \end{aligned}$$

If $\deg a_0(t) \geq 4$, we cannot kill the term of degree greater than or equal to 8 which comes from $a_0(t)^2$. Therefore, we can put $a_0(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3$. Then, we have equations:

$$\begin{aligned} c_1^2 &= c_1^7, \alpha_3^2 = 0, 0 = c_1 b_1^4, \alpha_2^2 + c_1 \alpha_3 = c_1^3 b_1, c_1 \alpha_2 = b_1^3 + c_1^3 b_0, \\ \alpha_1^2 + c_1 \alpha_1 &= b_0 b_1^2, c_1 \alpha_0 = c_1 b_0^4 + b_0^2 b_1, \alpha_0^2 = b_0^3. \end{aligned}$$

Solving these equations, we have

$$b_0 = 0, b_1 = 0, \alpha_0 = 0, \alpha_2 = 0, \alpha_3 = 0, c_1^5 = 1, \alpha_1 = 0 \text{ or } c_1.$$

Therefore, we have $c_1 = \zeta, e_1(t, x) = \zeta, a_0(t) = 0$ or $\zeta t, d_2(t) = 0$. with $\zeta^5 = 1, \zeta \in k$. Putting these data into the original equation, we have

$$\begin{aligned} \zeta^2(tx^4 + x^3 + t^3x) + a_1(t)^2x^2 + a_2(t)^2x^4 + \zeta ta_1(t)x + \zeta ta_2(t)x^2 \\ = \zeta td_1(t)^4x^4 + d_1(t)^3x^3 + \zeta^3 t^3 d_1(t)x. \end{aligned}$$

Considering the coefficients of x^4 , we have $\zeta^2 t + a_2(t)^2 + \zeta td_1(t)^4 = 0$. Therefore, we have $a_2(t) = 0$ and $d_1(t) = \zeta^4$. Considering the coefficients of x^2 , we have $a_1(t) = 0$. Therefore we have

$$c_1 = \zeta, \quad d_1(t) = \zeta^4, \quad d_2(t) = 0, \quad e_1(t, x) = \zeta, \quad e_2(t, x) = 0 \text{ or } \zeta t.$$

Fixing a fifth primitive root ζ of unity, we set

$$\begin{aligned} \sigma : t &\mapsto \zeta t, & x &\mapsto \zeta^4 x, & y &\mapsto \zeta y \\ \tau : t &\mapsto t, & x &\mapsto x, & y &\mapsto y + t. \end{aligned}$$

Then, we have

$$\sigma \circ \tau : t \mapsto \zeta t, \quad x \mapsto \zeta^4 x, \quad y \mapsto \zeta y + \zeta t$$

and $\langle \sigma \circ \tau \rangle \cong \mathbf{Z}/10\mathbf{Z}$. We now take the relative Jacobian variety of $f : X \rightarrow \mathbf{P}^1$. It has singular fibers of types IV, IV*, and the Mordell-Weil group is isomorphic to $\mathbf{Z}/3\mathbf{Z}$ (cf. Ito [9]). We denote by ρ a generator of the group. It acts on X and permutes three 2-sections. On the other hand, τ is induced from the action of the Mordell-Weil group $\mathbf{Z}/2\mathbf{Z}$ of a quasi-elliptic fibration p with singular fibers of type (III*, III) (cf. Ito [10]) and it interchanges two 2-sections not equal to the curve of cusps of p . Therefore, considering the action of the subgroup $\langle \tau, \rho \rangle$ generated by τ and ρ on the dual graph of (-2) -curves, we see $\langle \tau, \rho \rangle$ is isomorphic to the symmetric group \mathfrak{S}_3 of degree 3 which is not numerically trivial. Considering the commutation relations of σ, τ, ρ , we conclude $\text{Aut}(X) \cong \mathbf{Z}/5\mathbf{Z} \times \mathfrak{S}_3$ (see Subsection 4.2). The automorphism σ is numerically trivial by construction. \square

Remark 5.10. Note that $\text{Aut}_{ct}(X) = \text{Aut}_{nt}(X)$ because X is supersingular. The numerically trivial automorphism σ of order 5 is a new example of such automorphisms.

5.2. Classical case. We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \rightarrow \mathbf{P}^1$ associated with the Weierstrass equation

$$y^2 + xy + sy = x^3$$

with a parameter s . This surface is a rational elliptic surface with a singular fiber of type I_3 over the point given by $s = 0$, a singular fiber of type I_1 over the point given by $s = 1$ and a singular fibers of type IV^* over the point given by $s = \infty$ (cf. Lang [18, §2]). We consider the base change of $\psi : R \rightarrow \mathbf{P}^1$ by $s = t^2$. Then, we have the elliptic surface associated with the Weierstrass equation

$$(5.3) \quad y^2 + xy + t^2y = x^3.$$

We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$(5.4) \quad f : \tilde{R} \rightarrow \mathbf{P}^1.$$

The rational elliptic surface $f : \tilde{R} \rightarrow \mathbf{P}^1$ has a singular fiber of type I_6 over the point given by $t = 0$, a singular fiber of type I_2 over the point given by $t = 1$ and a singular fiber of type IV over the point given by $t = \infty$ (see Figure 6). The fibration f has six sections. In Figure 6, (-1) -curves denote the 0-section and the two sections defined by the equations

$$x = y = 0, \quad x = y + t^2 = 0$$

respectively.

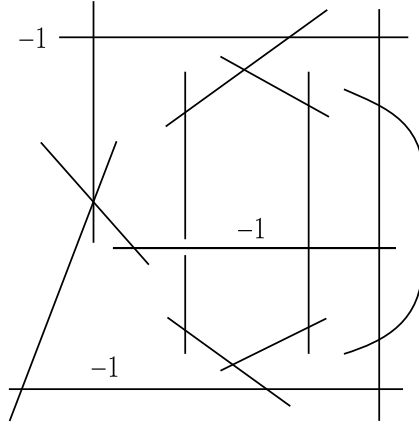


FIGURE 6

Now, we consider a rational vector field on \tilde{R} defined by

$$(5.5) \quad D = D_a = (t + a) \frac{\partial}{\partial t} + (x + t^2) \frac{\partial}{\partial x}$$

where $a \in k$, $a \neq 0, 1$. We see that $D^2 = D$, that is, D is 2-closed. Note that the nonsingular fiber F_a over the point defined by $t = a$ is integral with respect to D . The vector field D has an isolated singularity at the point P which is the singular point of the fiber of type IV. Denote by F_∞ , $E_{\infty,1}$ and $E_{\infty,2}$ the three components of the singular fiber of type IV. Then P is the intersection point of these three curves. To resolve this singularity, we first blow up at P . Denote by $E_{\infty,3}$ the exceptional curve. We denote the proper transforms of F_∞ , $E_{\infty,1}$ and $E_{\infty,2}$ by the same symbols. Then blow up at three points $E_{\infty,3} \cap (F_\infty + E_{\infty,1} + E_{\infty,2})$. Let Y be the obtained surface and $\psi : Y \rightarrow \tilde{R}$ the successive blowing-ups. We denote by the same symbol D the induced vector field on Y . We denote by $E_{\infty,4}$, $E_{\infty,5}$ or $E_{\infty,6}$ the exceptional curve over the point $E_{\infty,3} \cap F_\infty$, $E_{\infty,3} \cap E_{\infty,1}$ or $E_{\infty,3} \cap E_{\infty,2}$ respectively. Then we have the following Figure 7 in which we give the self-intersection numbers of the curves, and the thick curves are integral with respect to D .

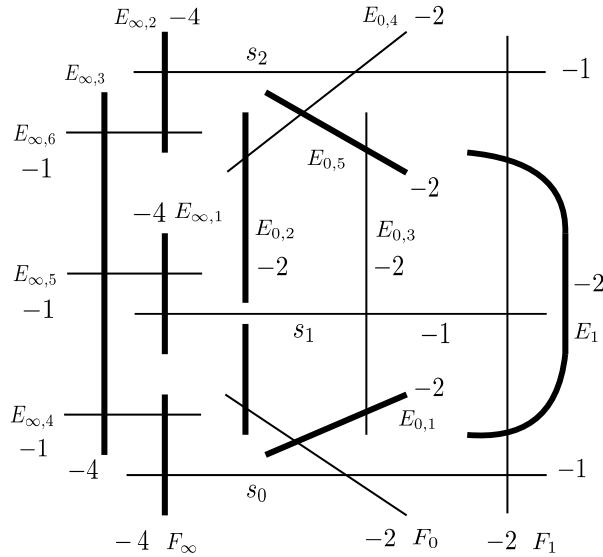


FIGURE 7

A direct calculation shows the following Lemmas.

Lemma 5.11. (i) *The divisorial part (D) of D on Y is given by*

$$-(E_1 + E_{0,1} + E_{0,2} + E_{0,5} + F_\infty + E_{\infty,1} + E_{\infty,2}) - 2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}).$$

(ii) *The integral curves in Figure 7 are*

$$E_{0,1}, E_{0,2}, E_{0,5}, F_\infty, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}, E_1.$$

Lemma 5.12. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by*

$$K_Y = -(F_\infty + E_{\infty,1} + E_{\infty,2}) - 2(E_{\infty,3} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6}).$$

(iii) $K_Y \cdot (D) = -4$.

Now, by taking the quotient by D , we have the following Figure 8. Here the numbers $-1, -4$ denote the self-intersection numbers of curves. The other curves have the self-intersection number -2 .

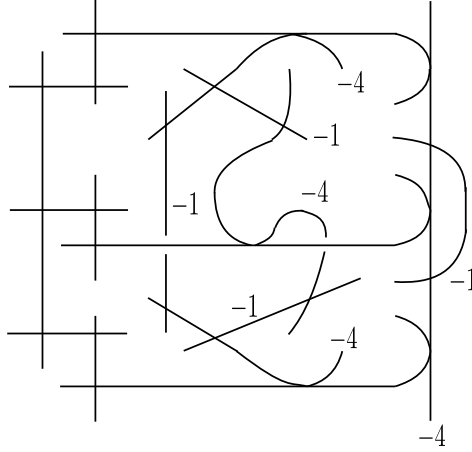


FIGURE 8

We now contract four (-1) -curves in Figure 8, and denote by X_a the obtained surface which has the dual graph of (-2) -curves given in Figure 5 (recall that the vector field (5.5) contains a parameter a). We use the notation of Figure 5. On X_a , there exist exactly one elliptic fibration with singular fibers of type $(2IV^*, I_3, I_1)$ defined by the linear system $|E_8 + E_9 + E_{10}|$ and three quasi-elliptic fibrations with singular fibers of type $(III^*, 2III)$ defined by $|2(E_9 + E_{12})|$, $|2(E_8 + E_{11})|$, $|2(E_{10} + E_{13})|$ respectively.

Theorem 5.13. *The surfaces $\{X_a\}$ form a 1-dimensional non-isotrivial family of classical Enriques surfaces with the dual graph given in Figure 5.*

Proof. By using Lemmas 5.11 and 5.12 and the same argument as in the case of the super-singular surface in the previous subsection, X_a is an Enriques surface. Since the image of

F_a and the singular fiber of type VI^* are double fibers, X_a is classical by Proposition 2.5. Moreover the double fiber F_a varies and hence this family is non-isotrivial. By the same proof as that of Theorem 5.8, we prove that X_a contains exactly 13 (-2) -curves whose dual graph is given in Figure 5. \square

Lemma 5.14. *The map $\rho_n : \text{Aut}(X_a) \rightarrow \text{O}(\text{Num}(X_a))$ is injective.*

Proof. Let $g \in \text{Ker}(\rho_n)$. Then g preserves each of the thirteen curves E_1, \dots, E_{13} (see Figure 5). First note that g fixes three points on each of E_8, E_9, E_{10} (in contrast to the supersingular case, where only two distinct points are fixed). Hence, g fixes E_8, E_9 and E_{10} pointwisely. Let p be the quasi-elliptic fibration with singular fibers of type $(\text{III}^*, 2\text{III})$ defined by the linear system $|2(E_8 + E_{11})|$ and let F be a general fiber of p . The two curves E_9, E_{10} are 2-sections of the fibration p . Then, g fixes at least three points on F which are the intersection with E_9 and E_{10} and the cusp of F . Hence, g fixes F pointwisely. Thus ρ_n is injective. \square

By the same arguments as in Theorems 5.8 and 5.9, we now have the following theorem.

Theorem 5.15. *The automorphism group $\text{Aut}(X_a)$ is isomorphic to the symmetric group \mathfrak{S}_3 of degree three and X_a contains exactly thirteen (-2) -curves.*

Proof. By Lemma 5.14, $\text{Aut}(X_a)$ is a subgroup of the symmetry group of the dual graph of (-2) -curves which is isomorphic to \mathfrak{S}_3 . By considering the actions of the Mordell-Weil groups of the Jacobian fibrations of genus one fibrations on X_a , any symmetry of the dual graph can be realized by an automorphism of X_a . \square

6. ENRIQUES SURFACES OF TYPE VII

The first and the second author proved the following theorem based on a method given in [12].

Theorem 6.1. ([13]) *There exists a 1-dimensional non-isotrivial family of Enriques surfaces with the dual graph of (-2) -curves given in Figure 9. A general member of this family is classical and a special member is supersingular. The automorphism group of any member in this family is isomorphic to the symmetric group \mathfrak{S}_5 of degree 5. The canonical cover of any member in this family has 12 ordinary nodes and its minimal resolution is the supersingular $K3$ surface with Artin invariant 1.*

There exist elliptic fibrations with singular fibers of type $(\text{I}_9, \text{I}_1, \text{I}_1, \text{I}_1)$, $(\text{I}_5, \text{I}_5, \text{I}_1, \text{I}_1)$, $(\text{I}_8, 2\text{III})$ or $(\text{I}_6, 2\text{IV}, \text{I}_2)$ on Enriques surfaces of type VII. For more details, we refer the reader to [13].

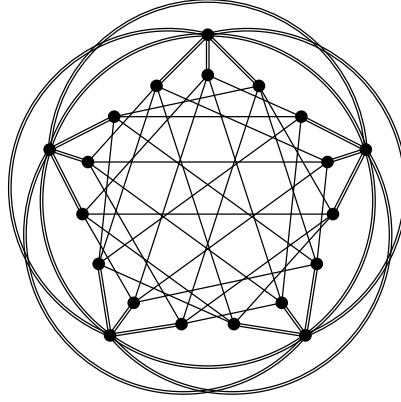


FIGURE 9

7. ENRIQUES SURFACES OF TYPE VIII

In this section we give a construction of a one-dimensional family of classical Enriques surfaces with the dual graph of type VIII.

We consider the relatively minimal nonsingular complete elliptic surface $\psi : R \longrightarrow \mathbf{P}^1$ associated with the Weierstrass equation

$$y^2 + sxy = x^3 + s^2x$$

with a parameter s . This surface is a rational elliptic surface with a singular fiber of type I_1^* over the point given by $s = 0$ and a singular fiber of type I_4 over the point given by $s = \infty$ (Lang [18, §2]). We consider the base change of $\psi : R \longrightarrow \mathbf{P}^1$ by $s = t^2$. Then, we have the Weierstrass model defined by

$$(7.1) \quad y^2 + txy + ty = x^3 + x^2$$

(see Lang [18, §2]). We consider the relatively minimal nonsingular complete model of this elliptic surface :

$$(7.2) \quad f : \tilde{R} \longrightarrow \mathbf{P}^1.$$

The rational elliptic surface $f : \tilde{R} \rightarrow \mathbf{P}^1$ has a singular fiber of type III over the point given by $t = 0$ and a singular fiber of type I_8 over the point given by $t = \infty$.

On the singular elliptic surface (7.1), we denote by F_0 the fiber over the point defined by $t = 0$, and by E_0 the fiber over the point defined by $t = \infty$. Both F_0 and E_0 are irreducible, and on each F_0 and E_0 , the surface (7.1) has only one singular point P_0 and P_∞ respectively. The surface \tilde{R} is a surface obtained by the minimal resolution of singularities

of (7.1). We use the same symbol for the proper transforms of curves on \tilde{R} . The blowing-up at the singular point P_0 gives one exceptional curve F_1 , and the surface is nonsingular along F_0 and F_1 . The two curves F_1 and F_0 make a singular fiber of type III of the elliptic surface $f : \tilde{R} \rightarrow \mathbf{P}^1$. On the other hand, the blowing-up at the singular point P_∞ gives two exceptional curves E_1, E_2 , and the surface is nonsingular along E_0 and has a unique singular point P_1 which is the intersection of E_1 and E_2 . The blowing-up at the singular point P_1 gives two exceptional curves E_3 and E_4 . The curves E_3 and E_4 meet at one point P_2 which is a singular point of the obtained surface. The blowing-up at the singular point P_2 again gives two exceptional curves E_5 and E_6 . The curves E_5 and E_6 meet at one point P_3 which is a singular point of the obtained surface. Finally, the blowing-up at the singular point P_3 gives an exceptional curve E_7 and the obtained surface is nonsingular over these curves. The cycle

$$E_0 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$$

forms a singular fiber of type I_8 given in Figure 10.

The elliptic surface $f : \tilde{R} \rightarrow \mathbf{P}^1$ has four sections s_i ($i = 0, 1, 2, 3$) given as follows:

$$\begin{aligned} s_0 &: \text{the zero section.} \\ s_1 &: x = y = 0. \\ s_2 &: x = t, y = 0. \\ s_3 &: x = 0, y = t. \end{aligned}$$

Also we consider the following two 2-sections b_1, b_2 defined by:

$$\begin{aligned} b_1 &: x + y = 0, x^2 + tx + t = 0. \\ b_2 &: x + y + tx + t = 0, x^2 + tx + t = 0. \end{aligned}$$

The configuration of singular fibers, three sections and two 2-sections is given in the following Figure 10:

Now, we consider a rational vector field on \tilde{R} defined by

$$D = D_a = t(at + 1) \frac{\partial}{\partial t} + (x + 1) \frac{\partial}{\partial x}, \quad a \neq 0 \in k.$$

Then, we have $D^2 = D$, that is, D is 2-closed. However, D has an isolated singularity at the point P which is the singular point of the fiber of type III, that is, the intersection point of two curves F_0 and F_1 (note that (x, t) is not a local parameter along F_0). To resolve this singularity, we first blow up at P . Denote by F_2 the exceptional curve. We denote the proper transforms of F_0 and F_1 by the same symbols. Then the induced vector field has three isolated singularities one of which is the intersection of three curves and other two points lie on the curve F_2 . Blow up at these three points. Let Y be the obtained surface and $\psi : Y \rightarrow \tilde{R}$ the successive blowing-ups. We denote the induced vector field by the same symbol D , and the four exceptional curves by F_2, F_3, F_4, F_5 . Then we have the following Figure 11.

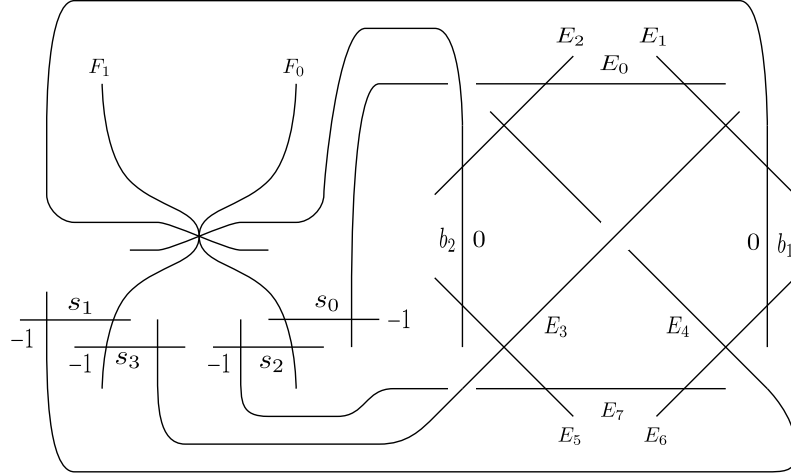


FIGURE 10

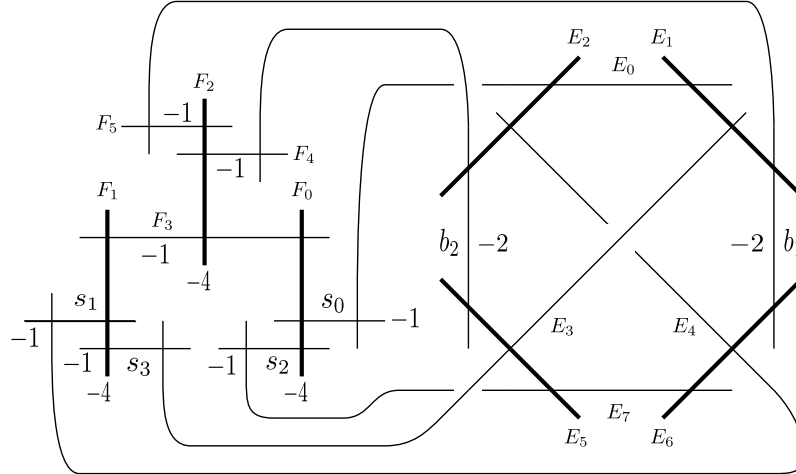


FIGURE 11

In the Figure 11 we give the self-intersection numbers of the curves except the curves with the self-intersection number -2 . Also the thick lines are integral curves with respect to D .

Denote by F_a the fiber over the point defined by $at = 1$. Then F_a is integral with respect to D . Now, according to the above blowing-ups, we see the following lemmas.

Lemma 7.1. (i) *The divisorial part (D) of the vector field D on Y is given by*

$$-(F_0 + F_1 + F_2 + 2F_3 + E_1 + E_2 + E_5 + E_6).$$

(ii) *The integral curves in Figure 11 are*

$$F_0, F_1, F_2, E_1, E_2, E_5, E_6.$$

Lemma 7.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by*

$$K_Y = -(F_0 + F_1 + F_2 + 2F_3).$$

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 12. In the Figure 12 we give the self-intersection numbers of the curves except the curves with the self-intersection number -2 .

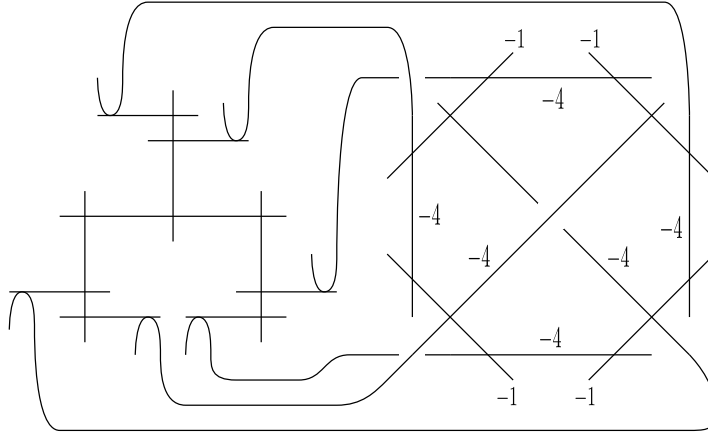


FIGURE 12

Let X_a be the surface obtained by contracting four exceptional curves in Figure 12 (Recall that the vector field D contains a parameter a). Then we have the following configuration of (-2) -curves in Figure 13.

The dual graph of the sixteen (-2) -curves in Figure 13 is nothing but the one given in Figure 14. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{D}_5 \oplus \tilde{A}_3$, $\tilde{D}_6 \oplus \tilde{A}_1 \oplus \tilde{A}_1$ or $\tilde{E}_6 \oplus \tilde{A}_2$.

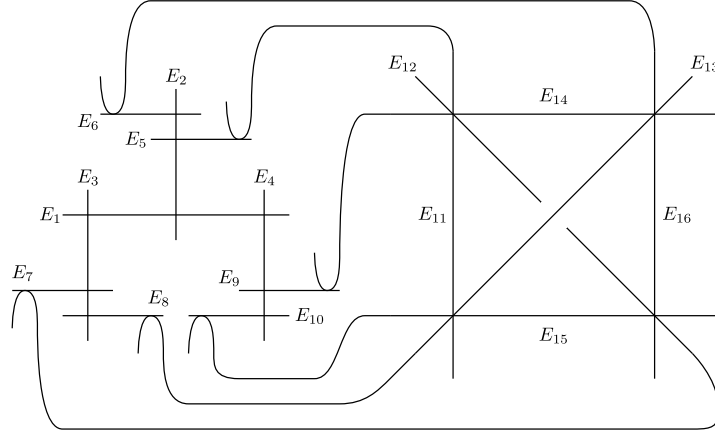


FIGURE 13

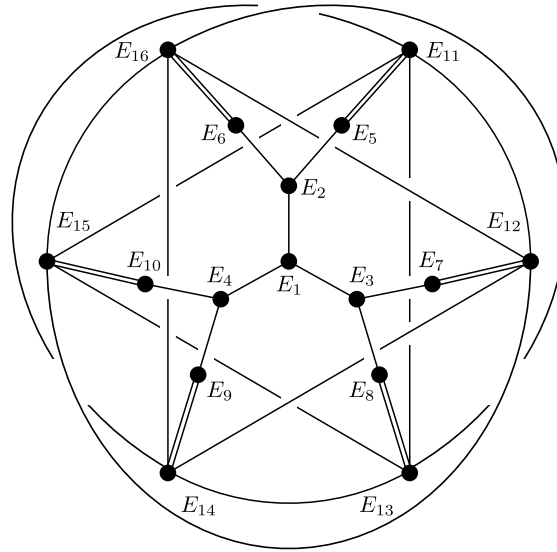


FIGURE 14

Theorem 7.3. *The surfaces $\{X_a\}$ form a non-isotrivial 1-dimensional family of classical Enriques surfaces with the dual graph given in Figure 14. The automorphism group $\text{Aut}(X_a)$ is isomorphic to \mathfrak{S}_4 .*

Proof. By using Lemmas 7.1 and 7.2 and the same argument as in the proof of Theorem 5.7, X_a is an Enriques surface. Since X_a has a quasi-elliptic fibration defined by $|2(E_5 + E_{11})| = |2(E_6 + E_{16})|$ with two double fibers, X_a is classical (Proposition 2.5). Note that the image of F_a is a double fiber of an elliptic fibration with singular fibers of type $(2I_1^*, I_4)$. Since F_a varies, this family is non-isotrivial. By the same proof as that of Theorem 5.8, X_a contains exactly 16 (-2) -curves whose dual graph given in Figure 14. The quasi-elliptic fibration defined by $|2(E_5 + E_{11})|$ has five 2-sections $E_2, E_{12}, E_{13}, E_{14}, E_{15}$. Each of these 2-sections meets another (-2) -curves at three different points, and hence they are fixed by any numerically trivial automorphisms. Therefore, by the same proof as that of Lemma 5.14, the natural map $\rho_n : \text{Aut}(X_a) \rightarrow \text{O}(\text{Num}(X_a))$ is injective. Note that the automorphism group of the dual graph is isomorphic to the symmetric group \mathfrak{S}_4 . By considering the actions of the Mordell-Weil groups of the Jacobian fibrations of genus one fibrations on X_a , we have proved that $\text{Aut}(X_a) \cong \mathfrak{S}_4$. \square

On X_a , there are three types of genus one fibrations: three elliptic fibrations with singular fibers of type $(2I_1^*, I_4)$, three quasi-elliptic fibrations with singular fibers of type $(I_2^*, 2\text{III}, 2\text{III})$ and eight elliptic fibrations with singular fibers of type (IV^*, I_3, I_1) .

8. ENRIQUES SURFACES OF TYPE \tilde{E}_8

In this section we give constructions of supersingular and classical Enriques surfaces with the following dual graph given of all (-2) -curves in Figure 15.



FIGURE 15

8.1. Supersingular case. Let (x, y) be an affine coordinate of $\mathbf{A}^2 \subset \mathbf{P}^2$. Consider a rational vector field D defined by

$$(8.1) \quad D = \frac{1}{x^5} \left((xy^6 + x^3) \frac{\partial}{\partial x} + (x^6 + y^7 + x^2y) \frac{\partial}{\partial y} \right)$$

Then $D^2 = 0$, that is, D is 2-closed. Note that D has a pole of order 5 along the line ℓ defined by $x = 0$ and this line is integral with respect to D . We see that D has a unique isolated singularity $(x, y) = (0, 0)$. First blow up at the point $(0, 0)$. Then we see that the exceptional curve is not integral and the induced vector field has a pole of order 2 along the exceptional curve. Moreover the induced vector field has a unique isolated singularity at the intersection of the proper transform of ℓ and the exceptional curve. Then continue this process until the induced vector field has no isolated singularities. The final configuration

of curves is given in Figure 16. Here F_0 is the proper transform of ℓ and the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

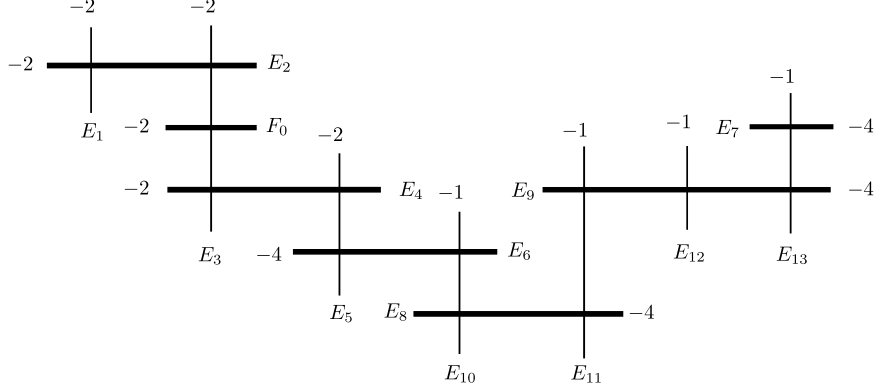


FIGURE 16

We denote by Y the surface obtained by this process. Also we denote by the same symbol D the induced vector field on Y . By direct calculations, we have the following lemmas.

Lemma 8.1. (i) *The integral curves with respect to D in Figure 16 are all horizontal curves (thick lines).*

(ii) $(D) = -(5F_0 + 2E_1 + 6E_2 + 8E_3 + 7E_4 + 4E_5 + 3E_6 + 2E_7 + 4E_8 + 5E_9 + 6E_{10} + 8E_{11} + 4E_{12} + 6E_{13})$.

Lemma 8.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(3F_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 4E_8 + 5E_9 + 6E_{10} + 8E_{11} + 4E_{12} + 6E_{13})$.*

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 17:

By contracting three exceptional curves, we get a new exceptional curve which is the image of the (-4) -curve meeting three exceptional curves. Let X be the surface obtained by contracting the exceptional curve. The surface X contains 10 (-2) -curves whose dual graph is given by Figure 15. Note that this diagram contains a unique maximal parabolic subdiagram which is of type \tilde{E}_8 . The pencil of lines in \mathbf{P}^2 through $(x, y) = (0, 0)$ induces a quasi-elliptic fibration on X with a double fiber of type Π^* .

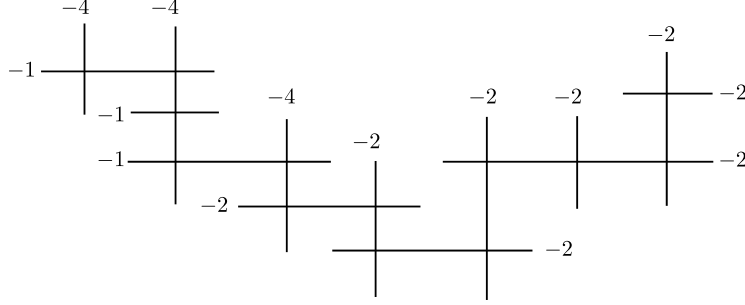


FIGURE 17

Theorem 8.3. *The surface X is a supersingular Enriques surface with the dual graph given in Figure 15.*

Proof. By using Lemmas 8.1 and 8.2 and the same arguments as in the proofs of Theorems 5.7 and 5.8, X is an Enriques surface with the dual graph given in Figure 15. Note that the normalization of the canonical cover of X is obtained from Y by contracting the divisor $F_0 + E_2 + E_3 + E_4$, and hence it has a rational double point of type D_4 . It follows from Lemma 12.6 (Section 12) that X is supersingular. \square

Theorem 8.4. $\text{Aut}(X) = \text{Aut}_{nt}(X) = \text{Aut}_{ct}(X) \cong \mathbf{Z}/11\mathbf{Z}$.

Proof. First note that the dual graph has no symmetries and hence $\text{Aut}(X) = \text{Aut}_{nt}(X)$. Since X is supersingular, $\text{Aut}_{ct}(X) = \text{Aut}_{nt}(X)$.

Now we consider the vector field (8.1), and we set $u = x^2$, $v = y^2$, $z = x^7 + xy^7 + x^3y$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^7 + uv^7 + u^3v$. Therefore, the quotient surface \mathbf{P}^2 by D is birationally isomorphic to the surface defined by $z^2 = u^7 + uv^7 + u^3v$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/u, \quad y = z/u^4, \quad t = v/u.$$

Then, the equation becomes $y^2 + tx^4 + x + t^7 = 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + x + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type II^* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2) -curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1.

Therefore, together with the equation $y^2 = tx^4 + x + t^7$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(tx^4 + x + t^7) + e_2(t, x)^2 \\ &= (c_1t + c_2)(d_1(t)x + d_2(t))^4 + (d_1(t)x + d_2(t)) + (c_1t + c_2)^7. \end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x , we have $e_1(t, x)^2 + d_1(t) = 0$. Therefore, $e_1(t, x)$ is a polynomial of t , i.e. we can put $e_1(t, x) = e_1(t)$, and $d_1(t) = e_1(t)^2$. Taking the coefficients of t , we have $e_1(t)^2x^4 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4 + d_2(t)_{\text{odd}}/t + c_1(c_1t + c_2)^6 = 0$. Here, $d_2(t)_{\text{odd}}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = c_1e_1(t)^8$. Since we have $e_1(t) \neq 0$, we have $e_1(t)^6 = 1/c_1$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = 1/c_1$. Therefore, we have an identity $e_1^2t^6 + c_1d_2(t)^4 + d_2(t)_{\text{odd}}/t + c_1(c_1t + c_2)^6 = 0$ with $e_1^6 = 1/c_1$. Let $d_2(t)$ be of degree m . If $m \geq 2$, then we have $\deg d_2(t)^4 \geq 8$ and we cannot kill the highest term of $d_2(t)^4$ in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ ($b_0, b_1 \in k$) and we have an identity

$$(e_1^2 + c_1^7)t^6 + (c_1b_1^4 + c_1^5c_2^2)t^4 + c_1^3c_2^4t^2 + (c_1b_0^4 + b_1 + c_1c_2^6) = 0.$$

Therefore, we have $e_1^2 + c_1^7 = 0$, $c_1b_1^4 + c_1^5c_2^2 = 0$, $c_1^3c_2^4 = 0$, $c_1b_0^4 + b_1 + c_1c_2^6 = 0$ with $e_1^6 = 1/c_1$. Since $c_1 \neq 0$, we have $c_2 = b_1 = b_0 = 0$ and $c_1 = \zeta$, $e_1 = \zeta^9$, $d_1 = \zeta^7$ with $\zeta^{11} = 1$. Putting this data into the original equation, we have $e_2(t, x) = 0$. These σ 's are really automorphisms of X and we conclude $\text{Aut}(X) \cong \mathbf{Z}/11\mathbf{Z}$ (see Subsection 4.2). \square

Remark 8.5. The automorphism σ is a new example of a cohomologically trivial automorphism.

8.2. Classical case. Let $Q = \mathbf{P}^1 \times \mathbf{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q . Let $x = u_0/u_1$, $x' = u_1/u_0$, $y = v_0/v_1$, $y' = v_1/v_0$. Consider a rational vector field D defined by

$$(8.2) \quad D = \frac{1}{x^3y^2} \left(x^4y^2 \frac{\partial}{\partial x} + (x^2 + ax^4y^4 + y^4) \frac{\partial}{\partial y} \right), \quad a \neq 0 \in k.$$

Then $D^2 = D$, that is, D is 2-closed. Note that D has a pole of order 3 along the divisor defined by $x = 0$, a pole of order 1 along the divisor defined by $x = \infty$ and a pole of order 2 along the divisor defined by $y = 0$. Moreover D has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of supersingular Enriques surfaces of type E_8 , we blow up the points of isolated singularities of D and those of the associated vector field, and finally get a vector field D , denoted by the same symbol, without isolated singularities. The configuration of curves is given in Figure 18. Here F_0 , E_1 , or E_2 is

A direct calculation shows the following two lemmas 8.6 and 8.7.

$$(ii) (D) = -(2F_0 + 3E_1 + E_2 + 2E_3 + 4E_5 + 4E_6 + 3E_7 + 2E_8 + 4E_9 + 5E_{10} + 6E_{11} + 8E_{12} + 4E_{13} + 6E_{14}).$$

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(2F_0 + 2E_1 + E_3 + 3E_5 + 4E_6 + 3E_7 + 2E_8 + 4E_9 + 5E_{10} + 6E_{11} + 8E_{12} + 4E_{13} + 6E_{14})$.*

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 19:

Let X_a be the surface obtained by contracting four exceptional curves in Figure 19 (Recall that the vector field D contains one parameter a (see (8.2))). Then X_a contains 10 (-2) -curves whose dual graph is given by Figure 15. Recall that this diagram contains a unique maximal parabolic subdiagram which is of type \tilde{E}_8 . The first projection from Q to \mathbf{P}^1 induces a quasi-elliptic fibration on X_a with two double fibers of type Π^* and of type Π .

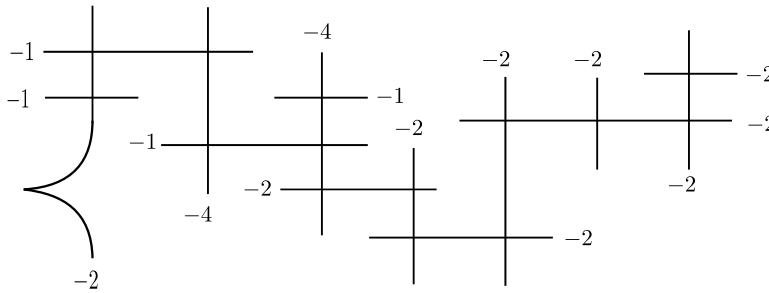


FIGURE 19

Theorem 8.8. *The surfaces $\{X_a\}$ form a 1-dimensional family of classical Enriques surfaces with the dual graph given in Figure 15.*

Proof. By using Lemmas 8.6 and 8.7 and the same arguments as in the proofs of Theorems 5.7 and 5.8, X_a is an Enriques surface with the dual graph given in Figure 15. Since X_a has a genus one fibration with two double fibers of type II^* , II , X_a is classical (Proposition 2.5). \square

Theorem 8.9. *The automorphism group $\text{Aut}(X_a)$ is trivial.*

Proof. We consider the vector field (8.2), and we set $u = x^2$, $v = y^2$, $z = x^3 + ax^5y^4 + xy^4 + x^4y^3$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^3 + a^2u^5v^4 + uv^4 + u^4v^3$ with $a \neq 0$. Therefore, the quotient surface $\mathbf{P}^1 \times \mathbf{P}^1$ by D is birationally isomorphic to the surface defined by $z^2 = u^3 + a^2u^5v^4 + uv^4 + u^4v^3$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/a^{\frac{3}{4}}uv, \quad y = z/a^{\frac{7}{4}}u^4v^2, \quad t = 1/\sqrt{a}u$$

and we replace $1/a^{\frac{5}{4}}$ by a for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x + t^3 + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type II^* exists over the point defined by $t = \infty$. Since σ preserves the dual graph of (-2) -curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the

point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1 t$.

Therefore, together with the equation $y^2 + tx^4 + at^3x + t^3 + t^7 = 0$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(tx^4 + at^3x + t^3 + t^7) + e_2(t, x)^2 \\ &= c_1 t(d_1(t)x + d_2(t))^4 + a(c_1 t)^3(d_1(t)x + d_2(t)) \\ & \quad + (c_1 t)^3 + (c_1 t)^7. \end{aligned}$$

Differentiating both sides by x , we have $ae_1(t, x)^2 t^3 + ac_1^3 d_1(t)t^3 = 0$, that is, $e_1(t, x)^2 = c_1^3 d_1(t)$. Therefore, $e_1(t, x)$ is a polynomial of t , i.e. we can put $e_1(t, x) = e_1(t)$, and $d_1(t) = c_1^{-3} e_1(t)^2$. Using Lemma 4.3 and taking the coefficients of t , we have $e_1(t)^2 x^4 + e_1(t)^2 t^2 + e_1(t)^2 t^6 + c_1(c_1^{-3} e_1(t)^2 x + d_2(t))^4 + ac_1^3 d_2(t)_{\text{even}} t^2 + c_1^3 t^2 + c_1^7 t^6 = 0$. Here, $d_2(t)_{\text{even}}$ is the even terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1^{-11} e_1(t)^8$. Since we have $e_1(t) \neq 0$, we have $e_1(t)^6 = c_1^{11}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = c_1^{11}$ and the equation becomes $e_1^2 t^2 + e_1^2 t^6 + c_1 d_2(t)^4 + ac_1^3 d_2(t)_{\text{even}} t^2 + c_1^3 t^2 + c_1^7 t^6 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1 t$ ($b_0, b_1 \in k$) and we have an identity

$$e_1^2 t^2 + e_1^2 t^6 + c_1(b_0 + b_1 t)^4 + ac_1^3 b_0 t^2 + c_1^3 t^2 + c_1^7 t^6 = 0.$$

Therefore, we have $e_1^2 = c_1^7$, $c_1 b_1^4 = 0$, $e_1^2 + ac_1^3 b_0 + c_1^3 = 0$ and $c_1 b_0^4 = 0$. Therefore, considering $e_1^6 = c_1^{11}$, we have $b_0 = b_1 = 0$, or $c_1 = e_1 = 1$. Therefore, we have $d_1(t) = 1$, $d_2(t) = 0$, $e_1(t, x) = 1$ and $e_2(t, x) = 0$. Hence, $\text{Aut}(X_a)$ is trivial. \square

9. ENRIQUES SURFACES OF TYPE $\tilde{E}_7 + \tilde{A}_1$

9.1. Classical case with a double fiber of type III*. In this subsection we give a construction of an Enriques surface with the following dual graph of all (-2) -curves given in Figure 20.



FIGURE 20

Let $(X_0, X_1, X_2) \in \mathbf{P}^2$ and $(S, T) \in \mathbf{P}^1$ be homogeneous coordinates. Consider the surface R defined by

$$(9.1) \quad S(aX_0^2 + bX_2^2) + T(X_1^2 + aX_1X_2 + bX_0X_2) = 0 \quad (a, b \in k, a \neq 0, b \neq 0).$$

Note that the projection to \mathbf{P}^1 defines a fiber space $\pi : R \rightarrow \mathbf{P}^1$ whose general fiber is a nonsingular conic. Let E_1 be the fiber over the point $(S, T) = (0, 1)$ which is nonsingular.

The fiber over the point $(S, T) = (1, 0)$ is a double line denoted by $2E_2$ and the fiber over the point (b^2, a^3) is a union of two lines denoted by E_3, E_4 . The line defined by $X_2 = 0$ is a 2-section of the fiber space which is denoted by F_0 . The surface R has two rational double points $Q_i = ((\alpha, \beta_i, 1), (1, 0))$ ($i = 1, 2$) of type A_1 , where $\alpha = \sqrt{b/a}$ and β_i is a root of the equation $y^2 + ay + \sqrt{b^3/a} = 0$.

Let $(x = X_0/X_2, y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

$$(9.2) \quad D = \frac{1}{s} \left(a(s^2 + c) \frac{\partial}{\partial x} + (as^2x^2 + bc) \frac{\partial}{\partial y} \right) \quad (b \neq a^2c)$$

where c is a root of the equation of $t^2 + (b/a)t + 1 = 0$. Then $D^2 = aD$, that is, D is 2-closed. A direct calculation shows that D has two isolated singularities at the intersection points of F_0 and E_1, E_2 . As in the case of supersingular Enriques surfaces of type E_8 , we blow up the two rational double points and the points of isolated singularities of D successively, and finally get a vector field, denoted by the same symbol D , without isolated singularities. The configuration of curves is given in Figure 21. Here, the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

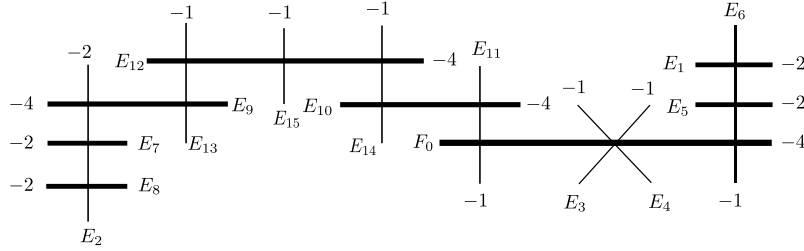


FIGURE 21

Now we denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

Lemma 9.1. (i) *The integral curves with respect to D in Figure 21 are all horizontal curves (thick lines).*

(ii) $(D) = -(F_0 + E_1 + 2E_2 + E_5 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 2E_{11} + 3E_{12} + 4E_{13} + 4E_{14} + 2E_{15})$.

Lemma 9.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(F_0 + 2E_2 + E_7 + E_8 + 2E_9 + 2E_{10} + 2E_{11} + 3E_{12} + 4E_{13} + 4E_{14} + 2E_{15})$.*

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 22:

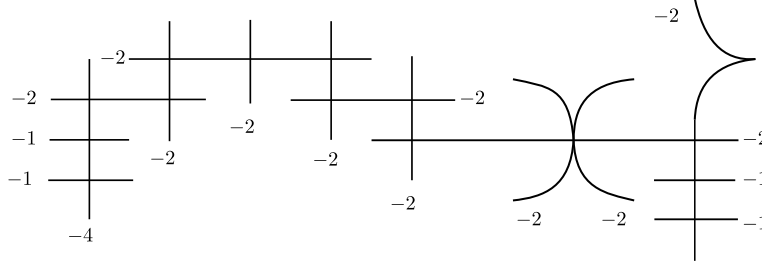


FIGURE 22

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves. The surface $X_{a,b}$ contains 11 (-2) -curves whose dual graph is given by Figure 20. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{E}_7 \oplus \tilde{A}_1$ or \tilde{E}_8 . On the surface $X_{a,b}$, there exist a quasi-elliptic fibration with singular fibers of type $(2\text{III}^*, \text{III})$ induced from the fiber space $\pi : R \rightarrow \mathbf{P}^1$ and two quasi-elliptic fibrations with a singular fiber of type II^* .

Theorem 9.3. *The surfaces $\{X_{a,b}\}$ form a 2-dimensional family of classical Enriques surfaces with the dual graph given in Figure 20. It contains an at least 1-dimensional, non-isotrivial family. The automorphism group $\text{Aut}(X_{a,b})$ is $\mathbf{Z}/2\mathbf{Z}$ which is not numerically trivial.*

Proof. By using Lemmas 9.1 and 9.2 and the same arguments as in the proofs of Theorems 5.7 and 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 20. Let p_1 be the genus one fibration with singular fibers $(2\text{III}^*, \text{III})$. By construction, p_1 has two double fibers (see Figure 22). Hence $X_{a,b}$ is classical (Proposition 2.5). In the next subsection 9.2, we give classical Enriques surfaces with double fibers of type III^* and III which are specializations of $\{X_{a,b}\}$. It follows from Matsusaka and Mumford [22, Theorem 1] that the family $\{X_{a,b}\}$ contains an at least 1-dimensional non-isotrivial family.

Next we determine the automorphism group. First we show that there are no numerically trivial automorphisms. Consider a genus one fibration p_2 with a singular fiber of type II^* . By using the classification of conductrices (Ekedahl and Shepherd-Barron [7], see also Table 5 in the later Section 12), p_2 is quasi-elliptic and the fiber of type II^* is simple. The simple component of the fiber of type III^* not meeting the special 2-section is the curve of

cusps of the fibration p_2 . Let C_1, C_2 be the double fibers of p_2 both of which are rational curves with a cusp. Let g be any numerically trivial automorphism. First assume that g is of order 2. Note that g preserves the double fiber C of type II of the fibration p_1 and g fixes two points on C which are the cusp of C and the intersection of C and the curve of cusps of p_1 . Hence g fixes C pointwisely. Since C is a 2-section of p_2 , C_i is preserved by g . Thus g fixes three points on C_i , which are the cusp of C_i and the intersection points of C_i with the two double fibers of p_1 , and hence g fixes C_1 and C_2 pointwisely. Therefore g fixes at least three points on a general fiber F of p_1 which are its cusp and the intersection points with C_1 and C_2 , and hence g fixes F pointwisely. Hence g is identity, that is, there are no numerically trivial automorphisms of even order. In the case where the order of g is odd, obviously, g preserves each C_i , and hence the above argument works well. Therefore, there are no numerically trivial automorphisms of $X_{a,b}$. Obviously the symmetry group of the dual graph of (-2) -curves is $\mathbf{Z}/2\mathbf{Z}$ (see Figure 20). By considering the action of the Mordell-Weil group of the Jacobian fibration of p_1 , we have $\text{Aut}(X_{a,b}) \cong \mathbf{Z}/2\mathbf{Z}$. \square

9.2. Classical case with double fibers of type III* and of type III. In this subsection we give a construction of classical Enriques surfaces with the following dual graph of all (-2) -curves given in Figure 23.



FIGURE 23

In the previous equations (9.1), (9.2), we set $b = 0$. Then $c = 1$ and the surface is defined by

$$(9.3) \quad aSX_0^2 + T(X_1^2 + aX_1X_2) = 0 \quad (a \in k, a \neq 0)$$

The fiber over the point $(S, T) = (0, 1)$ is a union of two lines, denoted by E_1, E_2 , defined by $X_1(X_1 + aX_2) = 0$. The fiber over the point $(S, T) = (1, 0)$ is a double line denoted by $2E_3$. The line defined by $X_2 = 0$ is a 2-section of the fiber space which is denoted by F_0 . The surface R has two rational double points $Q_1 = ((0, 0, 1), (1, 0))$, $Q_2 = ((0, a, 1), (1, 0))$ of type A_1 .

Let $(x = X_0/X_2, y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

$$(9.4) \quad D = \frac{1}{s} \left((s^2 + 1) \frac{\partial}{\partial x} + s^2 x^2 \frac{\partial}{\partial y} \right).$$

Then $D^2 = D$, that is, D is 2-closed. A direct calculation shows that D has two isolated singularities at the intersection points of the 2-section F_0 and two fibers over the points $(S, T) = (1, 0), (0, 1)$. As in the previous case, we blow up the two rational double points

and the points of isolated singularities of D successively, and finally get a vector field D , denoted by the same symbol, without isolated singularities. The configuration of curves is given in Figure 24.

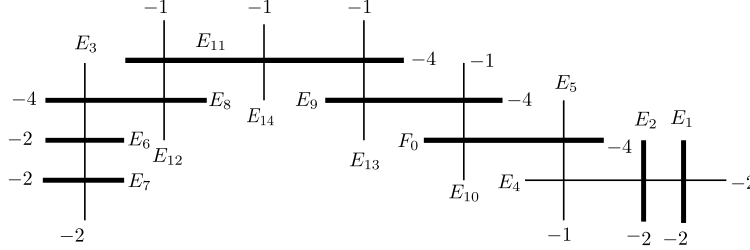


FIGURE 24

Here we use the same symbols F_0, E_1, E_2, E_3 for their proper transforms, and the suffix i of the other exceptional curve E_i corresponds to the order of successive blowing-ups. The thick lines are integral curves. We denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

Lemma 9.4. (i) *The integral curves with respect to D in Figure 24 are $F_0, E_1, E_2, E_6, E_7, E_8, E_9, E_{11}$ (thick lines).*

(ii) $(D) = -(F_0 + E_1 + E_2 + 2E_3 + 2E_6 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.

Lemma 9.5. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(F_0 + 2E_3 + E_6 + E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.*

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 25:

Let X_a be the surface obtained by contracting four exceptional curves. The surface X_a contains 11 (-2) -curves whose dual graph is given by Figure 23. Note that any maximal parabolic subdiagram of this diagram is of type $\tilde{E}_7 \oplus \tilde{A}_1$ or \tilde{E}_8 . The surface X_a has a quasi-elliptic fibration with singular fibers of type $(2\text{III}^*, 2\text{III})$ induced from the fiber space $\pi : R \rightarrow \mathbf{P}^1$ and a quasi-elliptic fibration with a singular fiber of type (II^*) .

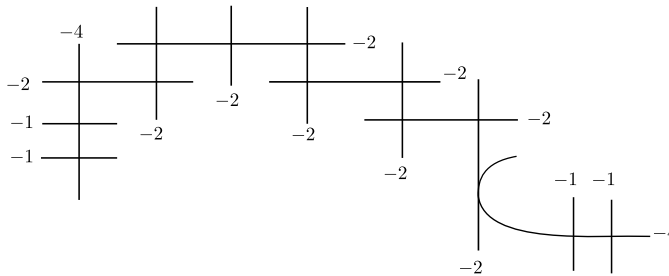


FIGURE 25

Theorem 9.6. *The surfaces $\{X_a\}$ form a 1-dimensional family of classical Enriques surfaces with the dual graph given in Figure 23. The automorphism group $\text{Aut}(X_a)$ is $\mathbf{Z}/2\mathbf{Z}$ which is numerically trivial.*

Proof. By using Lemmas 9.4 and 9.5 and the same arguments as in the proofs of Theorems 5.7 and 5.8, X_a is an Enriques surface with the dual graph given in Figure 23. Since X_a has a quasi-elliptic fibration with two double fibers, X_a is classical (Proposition 2.5). By the same argument as in the case of Theorem 9.3, we see that $|\mathrm{Aut}_{nt}(X_a)| \leq 2$. Since the dual graph of (-2) -curves on X_a has no symmetries (see Figure 23), we have $\mathrm{Aut}(X_a) = \mathrm{Aut}_{nt}(X_a)$. Let p be the quasi-elliptic fibration with singular fibers of type $(2\mathrm{III}^*, 2\mathrm{III})$. By considering the action of the Mordell-Weil group of the Jacobian fibration of p , we have $\mathrm{Aut}(X_a) \cong \mathbf{Z}/2\mathbf{Z}$. \square

9.3. Supersingular case with a double fiber of type III*. In this subsection we give a construction of supersingular Enriques surfaces with the dual graph of all (-2) -curves given in Figure 20.

Let $(X_0, X_1, X_2) \in \mathbf{P}^2$ and $(S, T) \in \mathbf{P}^1$ be homogeneous coordinates. Consider the surface R defined by

$$(9.5) \quad S(X_0^2 + a^3 X_2^2) + T(X_1^2 + X_1 X_2 + a^2 X_0 X_2) = 0 \quad (a \in k, a \neq 0).$$

Note that the projection to \mathbf{P}^1 defines a fiber space $\pi : R \rightarrow \mathbf{P}^1$ whose general fiber is a nonsingular conic. The fiber over the point $(S, T) = (a^4, 1)$ is a union of two lines denoted by E_1, E_2 and the fiber over the point $(S, T) = (1, 0)$ is a double line denoted by $2E_3$. The line defined by $X_2 = 0$ is a 2-section, denoted by F_0 , of the fiber space.

The surface R has two rational double points $Q_i = ((\alpha, \beta_i, 1), (1, 0))$ ($i = 1, 2$) where $\alpha = \sqrt{a^3}$ and β_i 's are roots of the equation $y^2 + y + a^3\sqrt{a} = 0$.

Let $(x = X_0/X_2, y = X_1/X_2, s = S/T)$ be an affine coordinate. Define

$$(9.6) \quad D = (s^2 + a) \frac{\partial}{\partial x} + (x^2 + a^2 s^2) \frac{\partial}{\partial y}.$$

Then $D^2 = 0$, that is, D is 2-closed. A direct calculation shows that D has an isolated singularity at the intersection point of the 2-section F_0 and the fiber over the point $(S, T) = (1, 0)$. As in the case of the previous section, we blow up the two rational double points and the point of isolated singularity of D successively, and finally get a vector field without isolated singularities. The configuration of curves is given in Figure 26.

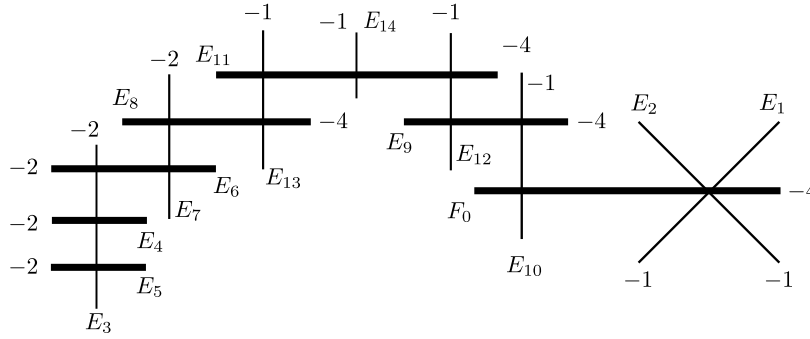


FIGURE 26

Here we use the same symbols F_0, E_1, E_2, E_3 for their proper transforms, and the suffix i of the other exceptional curve E_i corresponds to the order of successive blowing-ups.

We denote by Y the surface obtained by successive blowing-ups. By direct calculations, we have the following lemmas.

Lemma 9.7. (i) *The integral curves with respect to D in Figure 26 are all horizontal curves (thick lines).*

(ii) $(D) = -(F_0 + 4E_3 + 3E_4 + 3E_5 + 4E_6 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.

Lemma 9.8. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(F_0 + 2E_3 + E_4 + E_5 + 2E_6 + 2E_7 + 2E_8 + 2E_9 + 2E_{10} + 3E_{11} + 4E_{12} + 4E_{13} + 2E_{14})$.*

(iii) $K_Y \cdot (D) = -4$.

FIGURE 27

Theorem 9.9. *The surfaces $\{X_a\}$ form a 1-dimensional non-isotrivial family of supersingular Enriques surfaces with the dual graph given in Figure 20.*

Proof. By using Lemmas 9.7 and 9.8 and the same arguments as in the proofs of Theorems 5.7 and 5.8, X_a is an Enriques surface with the dual graph given in Figure 20. By construction, the normalization of the canonical cover has a rational double point of type D_4 . It now follows from Lemma 12.6 (Section 12) that X_a is supersingular. It follows from the following Theorem 9.10 and Matsusaka and Mumford [22, Theorem 1] that the family $\{X_a\}$ is non-isotrivial. \square

The surface X_a contains a unique quasi-elliptic fibration with singular fibers of type $(2\text{III}^*, \text{III})$ induced from the fiber space $\pi : R \rightarrow \mathbf{P}^1$ and two quasi-elliptic fibrations with a singular fiber of type (II^*) .

Theorem 9.10. *If $a^7 \neq 1$, then the automorphism group $\text{Aut}(X_a)$ is $\mathbf{Z}/2\mathbf{Z}$ which is not numerically trivial. If $a^7 = 1$, then the automorphism group $\text{Aut}(X_a)$ is $\mathbf{Z}/14\mathbf{Z}$ and $\text{Aut}_{nt}(X_a)$ is $\mathbf{Z}/7\mathbf{Z}$.*

Proof. We consider the vector field (9.6), and we set $T = s^2$, $u = x + as + s^3$ and $v = y + sx^2 + a^2s^3$. Here, $s = (y^2 + y + a^2x)/(x^2 + a^3)$ by (9.5). Then, we have $D(T) = 0$, $D(u) = 0$, $D(v) = 0$ with the equation $v^2 + v = Tu^4 + a^2u + T^7$ with $a \neq 0$ and the quotient surface \mathbf{P}^2 by D is birationally isomorphic to the surface defined by $v^2 + v = Tu^4 + a^2u + T^7$, which is birationally isomorphic to our Enriques surface. For the sake of simplicity, we replace a^2 by a . Then, the normal form becomes $v^2 + v = Tu^4 + au + T^7$. To calculate the automorphism group, we consider the change of coordinates with new coordinates x, y, t :

$$T = t + a^4, \quad v = y + a^2x^2 + ax, \quad u = x.$$

Then, the equation becomes $y^2 + y = tx^4 + (t + a^4)^7$ with $a \neq 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + y + tx^4 + (t + a^4)^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type III* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2) -curves, σ preserves $2F_\infty$. Therefore, σ preserves the structure of this quasi-elliptic surface. σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the point defined by $t = 0$ and σ preserves also the singular fiber. Therefore, we have $\sigma^*(t) = c_1t$.

Therefore, together with the equation $y^2 + y + tx^4 + (t + a^4)^7 = 0$, we have an identity

$$\begin{aligned} e_1(t, x)^2(y + tx^4 + (t + a^4)^7) + e_2(t, x)^2 + (e_1(t, x)y + e_2(t, x)) \\ = c_1t(d_1(t)x + d_2(t))^4 + (c_1t + a^4)^7. \end{aligned}$$

A is a free $k[t, x]$ -module, and 1 and y are linearly independent over $k[t, x]$. Taking the coefficient of y , we have $e_1(t, x)^2 + e_1(t, x) = 0$. Since $e_1(t, x) \neq 0$, we have $e_1(t, x) = 1$. Therefore, we have

$$\begin{aligned} tx^4 + (t + a^4)^7 + e_2(t, x)^2 + e_2(t, x) \\ = c_1t(d_1(t)x + d_2(t))^4 + (c_1t + a^4)^7. \end{aligned}$$

As a polynomial of x , if $e_2(t, x)$ has a term of degree greater than or equal to 3, then $e_2(t, x)^2$ has a term greater than or equal to 6. We cannot kill this term in the equation. By the equation, we know that $e_2(t, x)$ doesn't have terms of x of odd degree. Therefore, we can put $e_2(t, x) = a_0(t) + a_2(t)x^2$ with $a_0(t), a_2(t) \in k[t]$. We take the coefficients of x^4 . Then, we have $t + a_2(t)^2 + c_1td_1(t)^4 = 0$. Therefore, we have two equations $1 + c_1d_1(t)^4 = 0$ and $a_2(t)^2 = 0$. Therefore, we have $a_2(t) = 0$ and $d_1(t) = \frac{1}{\sqrt[4]{c_1}}$. The equation becomes $(t + a^4)^7 + a_0(t)^2 + a_0(t) = c_1td_2(t)^4 + (c_1t + a^4)^7$. Put $\deg d_2(t) = \ell$. Suppose $\ell \geq 2$. Then, the right-hand-side has an odd term whose degree is equal to $4\ell + 1 \geq 9$. Therefore, the left-hand-side must have an odd term which is of degree $4\ell + 1$. This means $\deg a_0(t) = 4\ell + 1$. However, in the equation we cannot kill the term of degree

$8\ell + 2$ which comes from $a_0(t)^2$. Therefore, we can put $d_2(t) = b_0 + b_1t$ with $b_0, b_1 \in k$. Then, the equation becomes

$$(t + a^4)^7 + a_0(t)^2 + a_0(t) = c_1b_0^4t + c_1b_1^4t^5 + (c_1t + a^4)^7$$

If $\deg a_0(t) \geq 4$, we cannot kill the term of degree greater than or equal to 8 in the equation which comes from $a_0(t)^2$. Therefore, we can put $a_0(t) = \alpha_0 + \alpha_1t + \alpha_2t^2 + \alpha_3t^3$. Then, we have equations:

$$\begin{aligned} 1 &= c_1^7, & a^4 + \alpha_3^2 &= c_1^6a^4, & a^8 &= c_1b_1^4 + c_1^5a^8, & a^{12} + \alpha_2^2 &= c_1^4a^{12}, \\ a^{16} + \alpha_3 &= c_1^3a^{16}, & a^{20} + \alpha_1^2 + \alpha_2 &= c_1^2a^{20}, \\ a^{24} + \alpha_1 &= c_1b_0^4 + c_1a^{24}, & a^{28} + \alpha_0^2 + \alpha_0 &= a^{28}. \end{aligned}$$

Assume $a^7 \neq 1$. Since $\alpha_3 = (c_1^3 + 1)a^2 = (c_1^3 + 1)a^{16}$, we have $(c_1^3 + 1)a^2(a^7 + 1)^2 = 0$. By $a^7 \neq 1$ and $a \neq 0$, we have $c_1^3 = 1$. Since $1 = c_1^7$, we have $c_1 = 1$. Therefore, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $b_0 = b_1 = 0$, and $\alpha_0 = 1$ or 0 . Therefore, we see that σ is given by either $t \mapsto t$, $x \mapsto x$, $y \mapsto y + 1$ or the identity. Hence, we have $\text{Aut}(X_a) \cong \mathbf{Z}/2\mathbf{Z}$ if $a^7 \neq 1$. Now, assume $a^7 = 1$. By $c_1^7 = 1$, c_1 is a seventh root of unity. We denote by ζ a primitive seventh root of unity. Then we have a solution

$$\begin{aligned} c_1 &= \zeta, \alpha_1 = 0, & \alpha_2 &= (1 + \zeta^2)a^6, & \alpha_3 &= (1 + \zeta^3)a^2, \\ b_0 &= \frac{(\sqrt[4]{\zeta} + 1)a^6}{\sqrt[4]{\zeta}}, & b_1 &= \frac{(\sqrt[4]{\zeta^5} + 1)a^2}{\sqrt[4]{\zeta}}. \end{aligned}$$

We have also $\alpha_0 = 1$ or 0 . Using this data, we have an automorphism σ which is defined by

$$\begin{aligned} t &\mapsto \zeta t \\ x &\mapsto \frac{1}{\sqrt[4]{\zeta}}x + \frac{(\sqrt[4]{\zeta} + 1)a^6}{\sqrt[4]{\zeta}} + \frac{(\sqrt[4]{\zeta^5} + 1)a^2}{\sqrt[4]{\zeta}}a^2t \\ y &\mapsto y + 1 + (1 + \zeta^2)a^6t^2 + (1 + \zeta^3)a^2t^3. \end{aligned}$$

This σ is of order 14, and by our argument the automorphism group is generated by σ . This means $\text{Aut}(X_a) \cong \mathbf{Z}/14\mathbf{Z}$ if $a^7 = 1$. By our construction, $\mathbf{Z}/7\mathbf{Z}$ is cohomologically (= numerically) trivial.

Finally we show that $\mathbf{Z}/2\mathbf{Z}$ is not numerically trivial. Assume that $g = \sigma^7$ is numerically trivial. Let p_1 be the quasi-elliptic fibration with singular fibers of type $(2\text{III}^*, \text{III})$ and let p_2 be a genus one fibration with singular fiber of type (II^*) . By using the classification of conductrices (Ekedahl and Shepherd-Barron [7], see also Table 5 in the later Section 12), we see that p_2 is quasi-elliptic and the fiber of type II^* is simple. Note that the simple component E of the singular fiber of type III^* not meeting the special 2-section is the curve of cusps of p_2 . Since g preserves the double fiber C of p_2 , g fixes two points on C which are the cusp of C and the intersection point of C and E . Thus g fixes C pointwisely. Obviously, g preserves a general fiber F of p_1 and fixes two points on F which are the cusp of F and the intersection with C . Hence g fixes F pointwisely. Thus we obtain $g = 1$ which is a contradiction. \square

Remark 9.11. The automorphism of order 7 is a new example of cohomologically trivial automorphisms.

10. ENRIQUES SURFACES OF TYPE \tilde{D}_8

In this section we give a construction of Enriques surfaces with the following dual graph of all (-2) -curves given in Figure 28.



FIGURE 28

10.1. Supersingular case. Let (x, y) be an affine coordinate of $\mathbf{A}^2 \subset \mathbf{P}^2$. Consider a rational vector field D defined by

$$(10.1) \quad D = D_a = \frac{1}{x^5} \left(x(x^4 + x^2 + y^6) \frac{\partial}{\partial x} + (ax^6 + y(x^4 + x^2 + y^6)) \frac{\partial}{\partial y} \right)$$

where $a \in k$, $a \neq 0$. Then $D^2 = 0$, that is, D is 2-closed. Note that D has poles of order 5 along the line ℓ defined by $x = 0$, and this line is integral. We see that D has a unique isolated singularity $(x, y) = (0, 0)$. First blow up at the point $(0, 0)$. Then we see that the exceptional curve is not integral and the induced vector field has poles of order 2 along the exceptional curve. Moreover the induced vector field has a unique isolated singularity at the intersection of the proper transform of ℓ and the exceptional curve. Continue this process until the induced vector field has no isolated singularities. The final configuration of curves is given in Figure 29. Here F_0 is the proper transform of ℓ and the suffix i of the exceptional curve E_i corresponds to the order of successive blowing-ups.

We denote by Y the surface obtained by this process. Also we denote by the same symbol D the induced vector field on Y . By direct calculations, we have the following lemmas.

Lemma 10.1. (i) *The integral curves with respect to D in Figure 29 are all horizontal curves (thick lines).*

$$(ii) (D) = -(5F_0 + 2E_1 + 6E_2 + 8E_3 + 7E_4 + 4E_5 + 3E_6 + 2E_7 + 2E_8 + 4E_9 + E_{10} + 2E_{11}).$$

Lemma 10.2. (i) $(D)^2 = -12$.

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(3F_0 + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 2E_8 + 4E_9 + E_{10} + 2E_{11})$.*

$$(iii) K_Y \cdot (D) = -4.$$

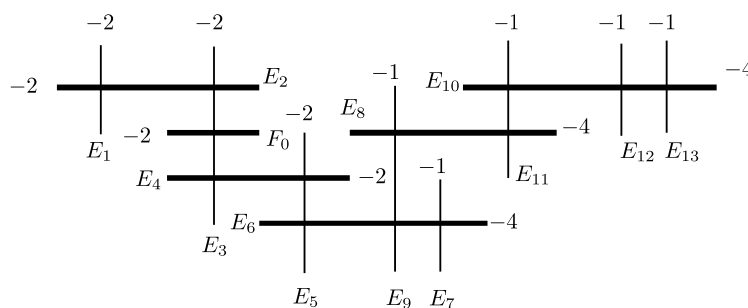


FIGURE 29

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 30.

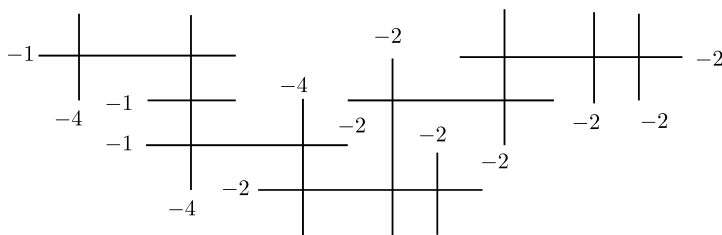


FIGURE 30

By contracting three exceptional curves, we get a new exceptional curve which is the image of the (-4) -curve meeting three exceptional curves. Let X_a be the surface obtained by contracting the new exceptional curve (Recall that the vector field (8.1) contains a parameter a). The surface X_a contains 10 (-2) -curves whose dual graph is given by Figure 28. Note that any maximal parabolic subdiagram of this diagram is of type \tilde{D}_8 or \tilde{E}_8 . On X_a there exist a quasi-elliptic fibration with singular fibers of type (I_4^*) induced from the pencil of lines in \mathbf{P}^2 through $(x, y) = (0, 0)$.

Theorem 10.3. *The surfaces $\{X_a\}$ form a 1-dimensional family of supersingular Enriques surfaces with the dual graph given in Figure 28.*

Proof. By using Lemmas 10.1 and 10.2 and the same arguments as in the proofs of Theorems 5.7 and 5.8, X is an Enriques surface with the dual graph given in Figure 28. By construction, the normalization of the canonical cover has a rational double point of type D_4 . Hence X_a is supersingular (Lemma 12.6, Section 12). \square

Remark 10.4. Note that X_a contains exactly three genus one fibrations. Let p_1 be the genus one fibration with a double singular fiber $2F_1$ of type I_4^* , and let p_i ($i = 2, 3$) be two genus one fibrations with a singular fiber F_i of type II^* . Note that p_1 is quasi-elliptic because its pull back to the canonical cover is a \mathbf{P}^1 -bundle, and p_2 and p_3 are elliptic because the conductrix is contained in the singular fiber of type II^* (see Lemma 12.2). Note that $F_1 \cdot F_2 = F_1 \cdot F_3 = F_2 \cdot F_3 = 2$. If both F_2 and F_3 are double fibers, then there are no canonical U -pairs on this Enriques surface which is a contradiction (Cossec and Dolgachev [3, Theorem 3.4.1]). Hence one of them, for example F_2 , is double and the other, F_3 , is simple. Since there are no automorphisms which change a double fiber and a simple fiber, any automorphism of X_a is cohomologically trivial.

Theorem 10.5. *The automorphism group $\text{Aut}(X_a)$ is the quaternion group Q_8 of order 8 which is cohomologically trivial.*

Proof. We consider the vector field (10.1), and we set $u = x^2$, $v = y^2$, $z = ax^7 + x^5y + x^3y + xy^7$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = a^2u^7 + u^5v + u^3v + uv^7$. Therefore, the quotient surface \mathbf{P}^2 by D is birationally isomorphic to the surface defined by $z^2 = a^2u^7 + u^5v + u^3v + uv^7$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = 1/u, \quad y = z/u^4, \quad t = v/u$$

and we replace a^2 by a for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + tx^2 + ax + t^7 = 0$. This equation gives a nonsingular affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + tx^2 + ax + t^7)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type I_4^* exists over the point defined by $t = \infty$. Since σ preserves the diagram of (-2) -curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1.

Therefore, together with the equation $y^2 = tx^4 + tx^2 + ax + t^7$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(tx^4 + tx^2 + ax + t^7) + e_2(t, x)^2 \\ &= (c_1t + c_2)(d_1(t)x + d_2(t))^4 + (c_1t + c_2)(d_1(t)x + d_2(t))^2 \\ & \quad + a(d_1(t)x + d_2(t)) + (c_1t + c_2)^7. \end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x , we have $ae_1(t, x)^2 + ad_1(t) = 0$. Therefore, $e_1(t, x)$ is a polynomial of t , i.e. we can put $e_1(t, x) = e_1(t)$, and $d_1(t) = e_1(t)^2$. Taking the coefficients of t , we have $e_1(t)^2x^4 + e_1(t)^2x^2 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4 + c_1(d_1(t)x + d_2(t))^2 + ad_2(t)_{\text{odd}}/t + c_1(c_1t + c_2)^6 = 0$. Here, $d_2(t)_{\text{odd}}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = c_1e_1(t)^8$. Since we have $e_1(t) \neq 0$, we have $e_1(t)^6 = 1/c_1$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = 1/c_1$. Considering the coefficients of x^2 , we have $e_1^2 = e_1(t)^2 = c_1d_1(t)^2 = c_1e_1^4$. Therefore, $e_1^2 = 1/c_1$. Therefore, we have $c_1 = 1$ and so $e_1 = d_1 = 1$. The equation becomes $t^6 + d_2(t)^4 + d_2(t)^2 + ad_2(t)_{\text{odd}}/t + (t + c_2)^6 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1t$ ($b_0, b_1 \in k$) and we have an identity

$$t^6 + (b_0 + b_1t)^4 + (b_0 + b_1t)^2 + ab_1 + (t + c_2)^6 = 0.$$

Therefore, we have $c_2 = b_1^2$, $c_2^2 = b_1$ and $b_0^4 + b_0^2 + ab_1 + c_2^6 = 0$. Therefore, we have either $c_2 = 0$, $b_1 = 0$, $b_0 = 0, 1$, or $c_2 = \omega$, $b_1 = \omega^2$ and $b_0 = \alpha$ is any root of $z^2 + z + \omega\sqrt{\alpha} + 1 = 0$. Here, ω is any cube root of unity. There exist 8 solutions. Putting these data into the original equation, we have $e_2(t, x) = \sqrt{\alpha}$ or $\omega^2x^2 + \omega^2x + \omega^2t^3 + \sqrt{\alpha\alpha} + \sqrt{\alpha}$. These σ 's are really automorphisms of X and we conclude $\text{Aut}(X) \cong Q_8$ (see Subsection 4.2). The cohomological triviality follows from Remark 10.4. \square

Remark 10.6. The group Q_8 is a new example of cohomologically trivial automorphisms.

10.2. Classical case. Let $Q = \mathbf{P}^1 \times \mathbf{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q . Let $x = u_0/u_1$, $x' = u_1/u_0$, $y = v_0/v_1$, $y' = v_1/v_0$. Consider a rational vector field D defined by

$$(10.2) \quad D = \frac{1}{xy^2} \left(ax^2y^2 \frac{\partial}{\partial x} + (x^4y^4 + by^4 + x^2y^2 + x^2) \frac{\partial}{\partial y} \right)$$

where $a, b \in k$, $a, b \neq 0$. Then $D^2 = aD$, that is, D is 2-closed. Note that D has a pole of order 1 along the divisor defined by $x = 0$, a pole of order 3 along the divisor defined by $x = \infty$ and a pole of order 2 along the divisor defined by $y = 0$. Moreover D has isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. As in the case of supersingular Enriques surfaces of type \tilde{E}_8 , we blow up the points of isolated singularities of D and those of associated vector fields, and finally get a vector field without isolated singularities. The configuration of curves is given in Figure 31.

Here F_0 , E_1 , or E_2 is the proper transform of the curve defined by $y = 0$, $x = 0$, or $x = \infty$, respectively.

We denote by Y the surface obtained by the successive blowing-ups. A direct calculation shows the following two lemmas.

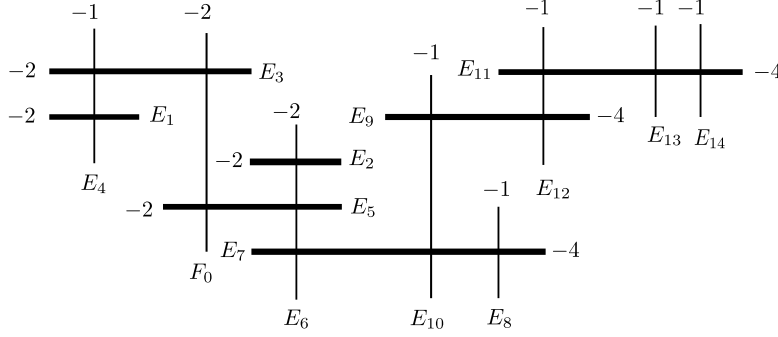


FIGURE 31

Lemma 10.7. (i) *The integral curves with respect to D in Figure 31 are all horizontal curves (thick lines).*

(ii) $(D) = -(2F_0 + E_1 + 3E_2 + 2E_3 + 4E_5 + 4E_6 + 3E_7 + 2E_8 + 2E_9 + 4E_{10} + E_{11} + 2E_{12}).$

Lemma 10.8. (i) $(D)^2 = -12.$

(ii) *The canonical divisor K_Y of Y is given by $K_Y = -(2F_0 + 2E_2 + E_3 + 3E_5 + 4E_6 + 3E_7 + 2E_8 + 2E_9 + 4E_{10} + E_{11} + 2E_{12}).$*

(iii) $K_Y \cdot (D) = -4.$

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and hence Y^D is nonsingular. By Proposition 2.1, we have the following configuration of curves in Figure 32.

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves in Figure 32 (Recall that the vector field D contains two parameters a, b (see (10.2))). On $X_{a,b}$, there exist 10 (-2) -curves whose dual graph is given by Figure 28. Recall that any maximal parabolic subdiagram of this diagram is of type \tilde{D}_8 or \tilde{E}_8 . On $X_{a,b}$ there exists a quasi-elliptic fibration with singular fibers of type (I_4^*) induced from the first projection from Q to \mathbf{P}^1 .

Theorem 10.9. *The surfaces $\{X_{a,b}\}$ form a 2-dimensional family of classical Enriques surfaces with the dual graph given in Figure 28.*

Proof. By using Lemmas 10.7 and 10.8 and the same arguments as in the proofs of Theorems 5.7 and 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 28. Since $X_{a,b}$ has a genus one fibration with two double fibers (see Figure 32), $X_{a,b}$ is classical (Proposition 2.5). \square

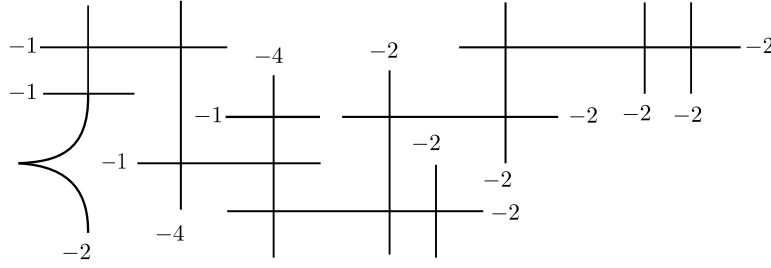


FIGURE 32

Remark 10.10. There are two genus one fibrations with a singular fiber of type II^* . As we explained in Remark 10.4, one of them is double and the other is simple. If its only singular fiber is (II^*) , then its j -invariant is zero (Lang [18]) and hence all nonsingular fibers are supersingular elliptic curves by Lemma 2.6. This contradicts the fact that a double fiber of a genus one fibration on a classical Enriques surface is an ordinary elliptic curve or an additive type (Proposition 2.4). Thus this fibration has singular fibers of type $(\text{II}^*, \text{I}_1)$ by Lang [18].

Theorem 10.11. *The automorphism group $\text{Aut}(X_{a,b})$ is $\mathbb{Z}/2\mathbb{Z}$ which is numerically trivial.*

Proof. It follows from Remark 10.10 that $\text{Aut}(X_{a,b}) = \text{Aut}_{nt}(X_{a,b})$. We consider the vector field (10.2), and we set $u = x^2$, $v = y^2$, $z = x^5y^4 + bxy^4 + x^3y^2 + x^3 + ax^2y^3$. Then, we have $D(u) = 0$, $D(v) = 0$, $D(z) = 0$ with the equation $z^2 = u^5v^4 + b^2uv^4 + u^3v^2 + u^3 + a^2u^2v^3$ with $a, b \neq 0$. Therefore, the quotient surface of $\mathbb{P}^1 \times \mathbb{P}^1$ by D is birationally isomorphic to the surface defined by $z^2 = u^5v^4 + b^2uv^4 + u^3v^2 + u^3 + a^2u^2v^3$, which is birationally isomorphic to our Enriques surface. To do a change of coordinates, we define new variables x, y, t by

$$x = \sqrt[4]{b}/uv, \quad y = \sqrt[4]{b^3}z/u^4v^2, \quad t = \sqrt{b}/u.$$

and we replace $\frac{1}{\sqrt{b}}$ and $\frac{a^2}{\sqrt[4]{b^5}}$ by a and b , respectively, for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0)$$

and let σ be an automorphism of our Enriques surface. The double fiber, denoted by $2F_\infty$, of type I_4^* exists over the point defined by $t = \infty$. Since σ preserves the dual graph of (-2) -curves, σ preserves the curve C of cusps and $2F_\infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, this quasi-elliptic surface has a singular fiber over the point defined by $t = 0$, σ preserves also the singular fiber. Therefore, we know $c_2 = 0$ and we have $\sigma^*(t) = c_1 t$.

Therefore, together with the equation $y^2 + tx^4 + at^3x^2 + bt^3x + t^3 + t^7 = 0$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(tx^4 + at^3x^2 + bt^3x + t^3 + t^7) + e_2(t, x)^2 \\ &= c_1 t(d_1(t)x + d_2(t))^4 + a(c_1 t)^3(d_1(t)x + d_2(t))^2 \\ & \quad + b(c_1 t)^3(d_1(t)x + d_2(t)) + (c_1 t)^3 + (c_1 t)^7. \end{aligned}$$

Differentiate both sides by x , and we have $be_1(t, x)^2 t^3 + bc_1^3 d_1(t)t^3 = 0$, that is, $e_1(t, x)^2 = c_1^3 d_1(t)$. Therefore, $e_1(t, x)$ is a polynomial of t , i.e. we can put $e_1(t, x) = e_1(t)$, and $d_1(t) = c_1^{-3} e_1(t)^2$. Using Lemma 4.3 and taking the coefficients of t , we have $e_1(t, x)^2(x^4 + at^2x^2 + t^2 + t^6) + c_1(c_1^{-3} e_1(t)^2 x + d_2(t))^4 + ac_1^3 t^2(c_1^{-3} e_1(t)^2 x + d_2(t))^2 + bc_1^3 d_2(t)_{\text{even}} t^2 + c_1^3 t^2 + c_1^7 t^6 = 0$. Here, $d_2(t)_{\text{even}}$ is the even terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1^{-11} e_1(t)^8$. Since we have $e_1(t) \neq 0$, we have $e_1(t)^6 = c_1^{11}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, we have $e_1^6 = c_1^{11}$. Considering the coefficients of x^2 of this equation, we have $ae_1^2 t^2 = ac_1^{-3} e_1^4 t^2$, i.e. $e_1^2 = c_1^3$. Therefore, we have $c_1^9 = c_1^{11}$. Since $c_1 \neq 0$, we have $c_1 = 1$. Therefore, we have $e_1 = 1$ and $d_1(t) = 1$. Then, the equation becomes $d_2(t)^4 + at^2 d_2(t)^2 + bd_2(t)_{\text{even}} t^2 = 0$. If the degree of $d_2(t)$ is greater than or equal to 2, then the highest term of $d_2(t)^4$ cannot be killed in the equation. Therefore, we can put $d_2(t) = b_0 + b_1 t$ ($b_0, b_1 \in k$) and we have an identity $(b_0 + b_1 t)^4 + a(b_0 + b_1 t)^2 t^2 + bb_0 t^2 = 0$. Therefore, we have $b_1^4 = ab_1^2$, $ab_0^2 = bb_0$ and $b_0^4 = 0$. Therefore, we have $b_0 = 0$, and $b_1 = \sqrt{a}$ or 0. Going to the original equality, we have $e_2(t, x)^2 = bt^3 \sqrt{at}$, i.e. $e_2(t, x) = \sqrt[4]{a} \sqrt{bt^2}$. Therefore, we conclude that σ is given by either $t \mapsto t$, $x \mapsto x + \sqrt{at}$, $y \mapsto y + \sqrt[4]{a} \sqrt{bt^2}$ or the identity. Hence, we have $\text{Aut}(X) \cong \mathbf{Z}/2\mathbf{Z}$. \square

11. ENRIQUES SURFACES OF TYPE $\tilde{D}_4 + \tilde{D}_4$

In this section we give a construction of Enriques surfaces with the following dual graph of all (-2) -curves given in Figure 33.

Let $Q = \mathbf{P}^1 \times \mathbf{P}^1$ be a nonsingular quadric and let $((u_0, u_1), (v_0, v_1))$ be a homogeneous coordinate of Q . Let $x = u_0/u_1$, $x' = u_1/u_0$, $y = v_0/v_1$, $y' = v_1/v_0$. Consider a rational vector field D defined by the equation (3.2):

$$D = \frac{1}{x^2 y^2} \left(bx^3 y^2 \frac{\partial}{\partial x} + (ax^2 y^2 + x^2 + x^4 y^4 + y^4 + bx^2 y^3) \frac{\partial}{\partial y} \right)$$

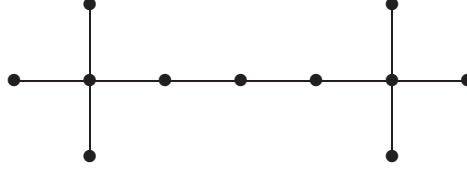


FIGURE 33

where $a, b \in k, b \neq 0$. Note that $D^2 = bD$, that is, D is 2-closed. Denote by E_1, E_2 and F_0 the curves defined by $x = 0, x' = 0$ and $y = 0$, respectively. The vector field D has poles of order 2 along E_1, E_2, E_3 , and has isolated singularities $(x, y) = (0, 0)$ and $(x', y) = (0, 0)$. The curves E_1, E_2 are integral. Now blow up at two points $(x, y) = (0, 0)$ and $(x', y) = (0, 0)$. The both exceptional curves are integral with respect to the induced vector field. The induced vector field has poles of order 3 along two exceptional curves and has isolated singularities at the intersections of the exceptional curves and the proper transforms of E_1 and E_2 . Then blow up at the isolated singularities of the induced vector field and continue this process until the induced vector field has no isolated singularities. We denote by Y the surface obtained by this process and by the same symbols E_1, E_2, F_0 their proper transforms. Also we denote by the same symbol D the induced vector field on Y . The final configuration of curves is given in Figure 34.

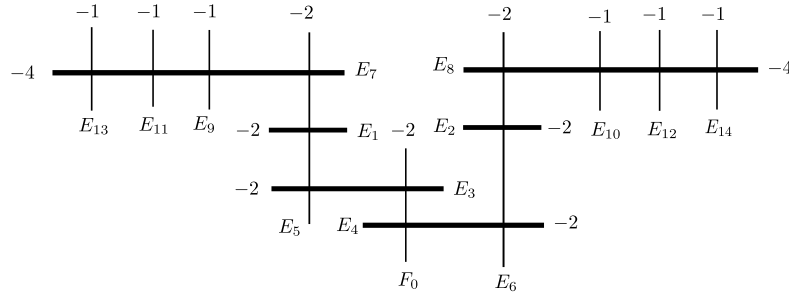


FIGURE 34

Lemma 11.1. (i) *The integral curves with respect to D in Figure 34 are all horizontal curves (thick lines).*

(ii) $(D) = -(2F_0 + 2E_1 + 2E_2 + 3E_3 + 3E_4 + 2E_5 + 2E_6 + E_7 + E_8)$.

Lemma 11.2. (i) $(D)^2 = -12$.

(ii) The canonical divisor K_Y of Y is given by $K_Y = -(2F_0 + E_1 + E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7 + E_8)$.

(iii) $K_Y \cdot (D) = -4$.

Now take the quotient Y^D of Y by D . By using the same argument as in the proof of Lemma 5.3, D is divisorial and Y^D is nonsingular. By Proposition 2.1, we have the following Figure 35.

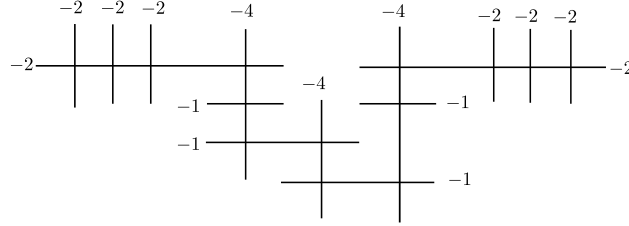


FIGURE 35

Let $X_{a,b}$ be the surface obtained by contracting four exceptional curves which contains 11 (-2) -curves whose dual graph is given by Figure 33. Note that any maximal parabolic subdaigram of this diagram is of type \tilde{D}_8 or $\tilde{D}_4 \oplus \tilde{D}_4$. The surface $X_{a,b}$ contains a quasi-elliptic fibration p_1 with singular fibers of type $(2I_0^*, 2I_0^*)$ induced from the first projection from Q to \mathbf{P}^1 and nine genus one fibrations with a singular fiber of type (I_4^*) . These nine genus one fibrations are elliptic by comparing to the conductrix given in Ekedahl and Shepherd-Barron [7, Theorem 2.2, Theorem 3.1] (see Tables 4 and 5 in the Section 12).

Theorem 11.3. *The surfaces $\{X_{a,b}\}$ form a 2-dimensional family of classical Enriques surfaces with the dual graph given in Figure 33. It contains an at least 1-dimensional, non-isotrivial family.*

Proof. By using Lemmas 11.1 and 11.2 and the same arguments as in the proofs of Theorems 5.7 and 5.8, $X_{a,b}$ is an Enriques surface with the dual graph given in Figure 33.

By (3.1) in Subsection 3.3, the surface $X_{a,b}$ is the quasi-elliptic surface given by the equation

$$u^2 + Sv^4 + a^2S^3v^2 + b^2S^4v + S^3 + S^7 = 0$$

By Queen [27, Theorem 2], its Jacobian is the quasi-elliptic surface given by

$$u^2 + Sv^4 + a^2S^3v^2 + b^2S^4v = 0$$

Now we change coordinates

$$Y = u/bS^2v^2, X = 1/v + a^2/b^2S, T = 1/S$$

which yields

$$Y^2 = X^3 + (a^4/b^4)T^2X + (1/b^2)T^3$$

Since these Jacobian quasi-elliptic surfaces form a 1-dimensional, non-isotrivial family by Ito [10], the family $\{X_{a,b}\}$ contains an at least 1-dimensional, non-isotrivial family. \square

Theorem 11.4. *The automorphism group $\text{Aut}(X_{a,b})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^3$. Moreover $\text{Aut}_{nt}(X_{a,b}) \cong (\mathbf{Z}/2\mathbf{Z})^2$.*

Proof. Equations of our classical Enriques surfaces of type $\tilde{D}_4 + \tilde{D}_4$ are given by (3.1) in Subsection 3.3. For our use, we set $x = v$, $y = u$, $t = S$ and we replace a^2 (resp. b^2) by a (resp. b) for the sake of simplicity. Then, the equation becomes $y^2 + tx^4 + at^3x^2 + bt^4x + t^3 + t^7 = 0$. This equation gives a normal affine surface. Set

$$A = k[t, x, y]/(y^2 + tx^4 + at^3x^2 + bt^4x + t^3 + t^7).$$

Our quasi-elliptic surface $\varphi : X \rightarrow \mathbf{P}^1$ has two double fibers of type I_0^* over the points defined by $t = 0$ (resp. $t = \infty$). First, we consider an automorphism τ defined by

$$\tau : t \mapsto 1/t, \quad x \mapsto x/t^2, \quad y \mapsto y/t^5.$$

This automorphism is of order 2 and exchanges two double fibers. Let σ be an automorphism of our Enriques surface. Then σ either keeps the double fibers or exchanges them. If σ exchanges the double fibers, then we consider $\tau \circ \sigma$. This keeps the double fibers. Therefore, we assume that σ keeps the double fibers. Since σ preserves the diagram of (-2) -curves, σ preserves the curve C of cusps and the double fiber $2F_\infty$ over $t = \infty$. Therefore, σ has the form in (4.4) in Subsection 4.1. Moreover, by our assumption, σ preserves the double fiber over the point defined by $t = 0$. Therefore, we may assume $\sigma^*(t) = c_1t$. Using these data, together with the equation $y^2 = tx^4 + at^3x^2 + bt^4x + t^3 + t^7$, we have an identity

$$\begin{aligned} & e_1(t, x)^2(tx^4 + at^3x^2 + bt^4x + t^3 + t^7) + e_2(t, x)^2 \\ &= c_1t(d_1(t)x + d_2(t))^4 + a(c_1t)^3(d_1(t)x + d_2(t))^2 \\ & \quad + b(c_1t)^4(d_1(t)x + d_2(t)) + (c_1t)^3 + (c_1t)^7. \end{aligned}$$

Using Lemma 4.3 and taking the coefficients of x , we have $be_1(t, x)^2t^4 + bc_1^4t^4d_1(t) = 0$. Therefore, we have $e_1(t, x)^2 + c_1^4d_1(t) = 0$ and $e_1(t, x)$ is a polynomial of t , i.e. we can put $e_1(t, x) = e_1(t)$, and $d_1(t) = e_1(t)^2/c_1^4$. Taking the coefficients of t , we have

$$\begin{aligned} & e_1(t)^2x^4 + ae_1(t)^2t^2x^2 + e_1(t)^2t^2 + e_1(t)^2t^6 + c_1(d_1(t)x + d_2(t))^4 \\ & + ac_1^3t^2(d_1(t)x + d_2(t))^2 + bc_1^4t^4d_2(t)_{\text{odd}}/t + c_1^3t^2 + c_1^7t^6 = 0. \end{aligned}$$

Here, $d_2(t)_{\text{odd}}$ is the odd terms of $d_2(t)$. Considering the coefficients of x^4 of this equation, we have $e_1(t)^2 = c_1d_1(t)^4 = e_1(t)^8/c_1^{15}$. Since we have $e_1(t) \neq 0$, we have $e_1(t)^6 = c_1^{15}$. Therefore, $e_1(t)$ is a constant and we set $e_1(t) = e_1 \in k$. Then, $e_1^6 = c_1^{15}$. Considering the

coefficients of x^2 , we have $at^2e_1^2 = ac_1^3t^2d_1(t)^2 = ac_1^3t^2(e_1^2/c_1^4)^2 = at^2e_1^4/c_1^5$. Therefore, $e_1^2 = c_1^5$ and $d_1(t) = c_1$. The equation becomes

$$e_1^2t^2 + e_1^2t^6 + c_1d_2(t)^4 + ac_1^3t^2d_2(t)^2 + bc_1^4t^4d_2(t)_{\text{odd}}/t + c_1^3t^2 + c_1^7t^6 = 0.$$

If $\deg d_2(t) \geq 2$, then we cannot kill the highest term of $c_1d_2(t)^4$ in the equation.

Therefore, we can put $d_2(t) = b_0 + b_1t$, and we have equations

$$e_1^2 = c_1^7, \quad c_1b_1^4 + ac_1^3b_1^2 + bc_1^4b_1 = 0, \quad e_1^2 + ac_1^3b_0^2 + c_1^3 = 0, \quad c_1b_0^4 = 0.$$

Solving these equations with $e_1^2 = c_1^5$, we have $b_0 = 0$, $c_1 = e_1 = d_1 = 1$, and b_1 is either 0 or a root of the equation $z^3 + az + b = 0$. Putting this data into the original equation, we have $e_2(t, x) = 0$. Hence, we have 4 automorphisms, which are the identity and three automorphisms of order 2. The involution τ and these automorphisms are commutative with each other. We now conclude $\text{Aut}(X) \cong (\mathbf{Z}/2\mathbf{Z})^3$ (see Subsection 4.2).

Obviously τ is not numerically trivial. We show that any involution σ preserving each double fiber of type I_0^* is numerically trivial. Let F be a double fiber of type I_0^* and let E be the component with multiplicity 2 of F . Then σ preserves E and a simple component C of F meeting with the special 2-section of the fibration, and hence it preserves one more simple component C' of F . This implies that σ fixes two points on E which are intersection points of E with C and C' . Therefore σ fixes E pointwisely and hence σ preserves all components of F . Thus σ is numerically trivial. \square

12. POSSIBLE DUAL GRAPHS

In this section, unless mentioned otherwise, all our Enriques surfaces are classical or supersingular.

12.1. Singularities of the canonical cover. In [7], Ekedahl and Shepherd-Barron studied "exceptional" Enriques surfaces using the conductrix associated to their canonical cover. In this section, we show that the non-normal locus as well as the isolated singularities of the canonical cover can be used to determine the dual graphs of (-2) -curves on Enriques surfaces with finite automorphism group. For this, we first need some preliminaries.

Lemma 12.1. (*Ekedahl and Shepherd-Barron [7, Lemma 0.9]*) *Let X be an Enriques surface, $\rho : \tilde{X} \rightarrow X$ its canonical cover and $\pi : X \rightarrow \mathbf{P}^1$ a genus one fibration. Then the morphism ρ factors through the pullback X_F of π by the Frobenius map on \mathbf{P}^1 . The map $\tilde{X} \rightarrow X_F$ is an isomorphism outside of the double fibers of π .*

Lemma 12.2. *Let X be an Enriques surface with conductrix A . Let π be a genus one fibration on X .*

- (1) *If π is a quasi-elliptic fibration, then the curve of cusps of π is a component of A with multiplicity 1.*
- (2) *If π is an elliptic fibration, then A is contained in one fiber of π .*

In particular, π is elliptic if and only if A is contained in a fiber of π .

Proof. A non-zero regular 1-form ω on X is given by the pullback of a regular 1-form on \mathbf{P}^1 (see [11]). Assume π is quasi-elliptic. Let F be a general cuspidal fiber and t a local parameter at $\pi(F)$. Then locally around the cusp of F is given by the equation $\pi^*t = y^2 + x^3$ (Bombieri-Mumford [1, Proposition 4]), hence $\omega = x^2 dx$ which vanishes twice at the cusp. Therefore, the curve of cusps is a component of A with multiplicity 1. Similarly one shows that ω does not vanish on any smooth point of an elliptic fiber of π if π is an elliptic fibration. Since A is connected, this yields the second claim. \square

Recall that the minimal dissolution of a double cover $Y \rightarrow X$ of surfaces with X smooth and Y normal is the successive blowing-ups of points on X lying under singular points of Y . For an Enriques surface X we call the minimal dissolution of the double cover $\tilde{X}_{norm} \rightarrow X$, where \tilde{X}_{norm} is the normalization of the canonical cover \tilde{X} , the minimal dissolution of X and denote it by X_{diss} . The normalization \tilde{X}_{sm} of X_{diss} in $K(\tilde{X})$ is the minimal resolution of singularities of \tilde{X}_{norm} if \tilde{X}_{norm} has only rational singularities.

Now, we recall the results of Ekedahl and Shepherd-Barron [7] on what happens to (-2) -curves on X when taking their inverse image in \tilde{X}_{sm} and additionally study curves of arithmetic genus 1.

Lemma 12.3. *With the notation introduced above, let C be an irreducible curve of arithmetic genus at most 1 on an Enriques surface X with conductrix A . Denote the irreducible curve on \tilde{X}_{sm} mapping surjectively to C by \tilde{C} and let $\rho : \tilde{X}_{sm} \rightarrow \tilde{X}$ and $\pi : \tilde{X} \rightarrow X$ be the morphisms from the normalization of the minimal dissolution of X to \tilde{X} and from \tilde{X} to X respectively. We fix the following invariants:*

- (i) *The degree s of $(\pi \circ \rho)|_{\tilde{C}} : \tilde{C} \rightarrow C$.*
- (ii) *The number r of points (including infinitely near ones) on C which are blown up during the minimal dissolution of X and their multiplicity m .*
- (iii) *The intersection number $A \cdot C$.*
- (iv) *The self-intersection numbers \tilde{C}^2 and C^2 .*
- (v) *The arithmetic genera $p_a(C)$ and $p_a(\tilde{C})$.*
- (vi) *If $p_a(C) = 1$, the type Sing of singularity of C . This is either nodal n , cuspidal c or smooth sm .*

Then \tilde{C} satisfies the following:

- (1) $\tilde{C}^2 = (C^2 - m^2 r) s^2 / 2$ and $2p_a(\tilde{C}) - 2 = \tilde{C}^2 - s A \cdot C$
- (2) *If two curves meet transversally on X and both have s -invariant 1, then they do not meet on X_{diss} .*
- (3) *For $A \cdot C \geq -2$ and $p_a(C) = 0$, we have the following possibilities*

r	s	$A \cdot C$	\tilde{C}^2	$p_a(\tilde{C})$
0	1	1	-1	0
0	2	-1	-4	0
2	1	0	-2	0
4	1	-1	-3	0
6	1	-2	-4	0
1	2	-2	-6	0

(4) For $p_a(C) = 1$, we have the following possibilities

Sing	r	m	s	$A \cdot C$	\tilde{C}^2	$p_a(\tilde{C})$
sm	0		1	0	0	1
sm	0		2	0	0	1
n	1	2	1	0	-2	0
c	0		1	0	0	1
c	0		2	0	0	1
c	1	2	1	0	-2	0
c	4	1	1	0	-2	0
c	2	1	1	1	-1	0
c	0		1	2	0	0

(5) If C is a cuspidal curve such that

- $|C|$ defines a quasi-elliptic fibration, then $r = 0$ and $s = 1$
- $|C|$ defines an elliptic fibration, then $r = 1$, $m = 2$ and $s = 1$
- $|C|$ does not define a quasi-elliptic fibration and $|2C|$ defines a quasi-elliptic fibration, then $r = 2$, $m = 1$ and $s = 1$.

Proof. Similar to Ekedahl and Shepherd-Barron [7], the formulas for the self-intersection number and the genus of \tilde{C} are obtained by observing that the self-intersection number of C drops by m^2 for every point of multiplicity m on C which is blown up during the minimal dissolution and from $\omega_{\tilde{X}/X} = \pi^*(\mathcal{O}_X(-A))$. Also, the claim (2) is in [7].

The first table is contained in [7] and we will only establish the second one. Therefore, assume that $p_a(C) = 1$. If C is smooth, then $A \cdot C = 0$ by Lemma 12.2 which only leaves the two possibilities listed. If C has a node, then $|C|$ defines an elliptic fibration φ with C as a simple fiber. Therefore, formally locally around C , X is isomorphic to the Jacobian of φ and by Lemma 12.1 we can find \tilde{C} by doing Frobenius pullback along the base. But on an I_1 fiber, an elliptic surface acquires an A_1 -singularity at the singular point of the nodal curve after Frobenius pullback. Therefore, the node of C is blown up during the minimal dissolution. A similar argument works if C is cuspidal and $|C|$ defines an elliptic fibration.

If C is cuspidal, we have enumerated all numerical possibilities except for the ones where $p_a(\tilde{C}) = 0$ and $s = 2$. These cases do not occur. In fact, assume that $s = 2$ and $p_a(\tilde{C}) = 0$. Denote the image of \tilde{C} on \tilde{X}_{norm} by \tilde{C}' . Since the singular point of C is not

blown up during the dissolution (by the self-intersection formula), we have $\tilde{C}' \cong \tilde{C} \cong \mathbf{P}^1$. Then, the flat morphism $\varphi : \tilde{X}_{\text{norm}} \rightarrow X$ restricts to a morphism $\varphi|_{\tilde{C}'} : \tilde{C}' \rightarrow C$. Since $s = 2$, we have $\varphi^*C = \tilde{C}'$ so $\varphi|_{\tilde{C}'}$ is nothing but the base change of φ along the closed immersion $C \rightarrow X$ and as such it is a flat morphism. But a morphism from \mathbf{P}^1 to the cuspidal cubic is never flat.

For the last statement (5), observe that $|C|$ defines a quasi-elliptic fibration if and only if $A \cdot C = 2$, and $|2C|$ defines a quasi-elliptic fibration if and only if $A \cdot C = 1$. This follows immediately from Lemma 12.2, which implies that $A \cdot C = D \cdot C$ where D is the curve of cusps of $|C|$ (resp. $|2C|$). \square

Remark 12.4. Several of the numerical possibilities in Lemma 12.3 might be excluded by using Lang's list of possible configurations of singular fibers on rational elliptic surfaces in characteristic 2 [19] together with Lemma 12.1. However, we will not pursue this here.

Lemma 12.5. *Let X be an Enriques surface with a quasi-elliptic fibration φ . Let F be a fiber of φ . If F is a double fiber, then two points on F (including infinitely near ones) are blown up during the minimal dissolution. If F is simple, then no point on F is blown up.*

Proof. If F is reducible, this can be read off from the table in [7, p.13], since every (-2) -curve on a simple fiber has r -invariant 0 and exactly one (-2) -curve on a double fiber has r -invariant 2 while the others have r -invariant 0. If F is irreducible, this is the last statement of Lemma 12.3. \square

Corollary 12.6. *Let X be an Enriques surface with a quasi-elliptic fibration. Then the normalization \tilde{X}_{norm} of the canonical cover has an isolated D_4 -singularity if and only if X is supersingular.*

Proof. Let φ be a quasi-elliptic fibration on X . Since the conductrix is non-empty by Lemma 12.2, \tilde{X} is not normal. Therefore, \tilde{X}_{norm} has either four A_1 - or one D_4 -singularity by Proposition 2.14. If φ has two double fibers, at least two distinct points on X are blown up during the minimal dissolution by Lemma 12.5. In this case, X is classical (Proposition 2.5) and \tilde{X} has four A_1 -singularities. If φ has only one double fiber, at most two distinct points on X are blown up. In this case, X is supersingular and \tilde{X} has one D_4 -singularity. \square

12.2. Special extremal genus one fibrations. In this section, we present a detailed study of Enriques surfaces with special genus one fibrations, their conductrices and isolated singularities on their canonical cover. Throughout, we will use the observations summed up in the following Lemma.

Lemma 12.7. *Let X be an Enriques surface with a conductrix A and \tilde{X} its canonical cover. The following hold.*

- (1) *If two (-2) -curves which meet transversally have s -invariant 1, then their intersection is blown up.*
- (2) *Every (-2) -curve meets the conductrix at most once.*
- (3) *Every (-2) -curve which is not a component of the conductrix has s -invariant 1.*

Now let $\pi : X \rightarrow \mathbf{P}^1$ be a genus one pencil. Then the following hold.

- (a) *A singular fiber of type I_n of π gives n A_1 -singularities on \tilde{X} .*
- (b) *If $A \neq \emptyset$ and π has a singular fiber of type I_n , then \tilde{X} has four A_1 -singularities.*
- (c) *If $A \neq \emptyset$ and two disjoint (-2) -curves have positive r -invariant, then \tilde{X} has four A_1 -singularities.*
- (d) *If $A \neq \emptyset$ and the sum of all r -invariants of fiber components is less than 4, then \tilde{X} has one D_4 -singularity.*

Proof. The first claim is obtained by checking intersection numbers, as was done by Ekedahl and Shepherd-Barron in [7] and the second is a consequence of Lemma 12.3. Since a curve C which is not contained in A has $A \cdot C \geq 0$, the third claim follows from Lemma 12.3.

For the statements about π : The first can be checked using the Jacobian of π , since an I_n fiber is simple. The second claim follows immediately from the first, since \tilde{X} has either four A_1 -singularities or one D_4 -singularity if $A \neq \emptyset$ (see Proposition 2.14). Two disjoint curves having positive r -invariant means that distinct points are blown up during the dissolution, excluding the possibility of a D_4 -singularity on the cover. For the last claim, the sum of r -invariants of fiber components being less than 4 means that less than 4 distinct points are blown up, so the singularity can only be a D_4 -singularity. \square

Remark 12.8. Observe that we have used that the singularities lying over a simple fiber of π can be read off from the Frobenius base change of the Jacobian fibration.

Lemma 12.9. *There are no special elliptic fibrations on Enriques surfaces with a double fiber of type 2III^* , 2II^* or 2I_4^* . Moreover, if the conductrix is nonempty, a special elliptic fibration with a double fiber of type IV can not exist.*

Proof. The statement about II^* , III^* and I_4^* is contained in Ekedahl and Shepherd-Barron [7, Corollary 3.2]. We will give another argument here. Let N be a special 2-section and C the simple component of the double fiber we want to exclude. By checking all possible conductrices of [7, Theorem 3.1], we obtain that C and N have s -invariant 1. Moreover, $A \cdot C = 0$ if C is a component of A with multiplicity 1, whereas $A \cdot C = 1$ if C does not occur in the conductrix. Therefore, $N \cdot A = 1$ if and only if $C \cdot A = 0$. Now by Lemma 12.7 (1), the intersection of N and C is blown up. But one of them has r -invariant 0 by Lemma 12.3. This is a contradiction.

Now, we prove the second claim. Since N has s -invariant 1 by Lemma 12.7 (3) and every component of the fiber of type IV also has s -invariant 1 by the same Lemma, the

intersection of N and the fiber of type IV is blown up. Additionally, the intersection of the three components of the fiber of type IV is blown up. Therefore, the canonical cover has four A_1 -singularities by Proposition 2.14. But every component of the fiber of type IV and N have r -invariant 2. This can not be achieved by blowing-ups at only 4 distinct points. \square

Lemma 12.10. *The isolated singularities on the normalization of the canonical cover of an Enriques surface with a special extremal elliptic fibration and the conductrix are summed up in table 4. The self-intersection number of the reduced inverse image of the curve on the minimal resolution of singularities of the canonical cover is given as an index to the multiplicity.*

Proof. For the list of rational extremal elliptic fibrations see Proposition 2.7. We will use the tables in [7, p.16-18] for the possibilities of the conductrix A . In every case, we denote the special 2-section by N . Recall that $A^2 = -2$ by Proposition 2.14.

- (I_4^*) : There is only one possibility for A with $A^2 = -2$. The canonical cover has four A_1 -singularities by Lemma 12.7 (c).
- (II^*) : There are two possible conductrices with $A^2 = -2$. However, since $N \cdot A \leq 1$ by Lemma 12.7 (2), we get the one in the table. Since all fibers different from the fiber of type II^* are smooth and no point on a smooth fiber is blown up during the dissolution by Lemma 12.3, the sum of all r -invariants of fibers is less than 4. Hence the cover has one D_4 -singularity by Lemma 12.7 (d).
- $(2III, I_8)$: In this case $A = \emptyset$. Since the intersection of N with a component of the fiber of type III is blown up, there are at least 11 distinct points which are blown up during the dissolution by Lemma 12.7 (a). Therefore, the cover has 12 A_1 -singularities.
- (III, I_8) : Again, we have $A = \emptyset$. By [19], the fiber of type III acquires a D_4 -singularity after Frobenius pullback. The 8 A_1 -singularities come from the fiber of type I_8 by Lemma 12.7 (a).
- $(2I_1^*, I_4)$: By Lemma 12.7 (b), we have 4 A_1 -singularities. Since every point which is blown up lies on the fiber of type I_4 , the r -invariant of N is at most 1 and therefore $N \cdot A = 1$. This is only possible for the conductrix in our table.
- (I_1^*, I_4) : By the same argument as in the previous case, we have $N \cdot A = 1$. Moreover, N can not meet distinct components of the fiber of type I_1^* since we would obtain a different fibration with a double fiber of type I_4 or I_5 in these cases. Therefore, N meets a multiplicity 2 component of the fiber of type I_1^* . Now, N and some components of the fiber of type I_1^* form a fiber of type I_0^* of a different fibration and the only possible conductrix for this behaviour is the one in our table.
- (III^*, I_2) : There are two possible conductrices with $A^2 = -2$. If the conductrix has the full fiber as support, N meets the central multiplicity 2 component since $N \cdot A \leq 1$ by Lemma 12.7 (2). But then, there is a fiber of type IV^* of a different

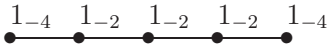
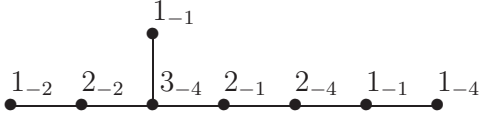
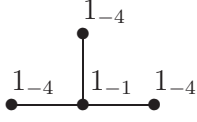
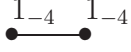
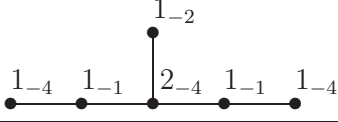
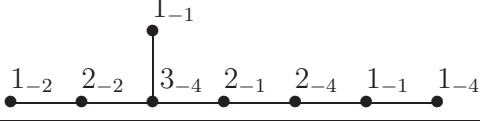
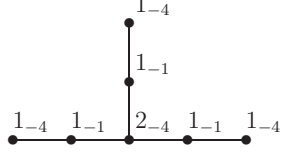
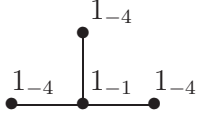
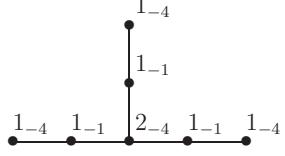
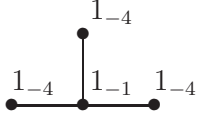
Singular fibers	Conductrix	Isolated singularities
(I_4^*)		$4A_1$
(II^*)		D_4
$(2III, I_8)$	\emptyset	$12A_1$
(III, I_8)	\emptyset	$D_4, 8A_1$
$(2I_1^*, I_4)$		$4A_1$
(I_1^*, I_4)		$4A_1$
(III^*, I_2)		$4A_1$
(II^*, I_1)		$4A_1$
$(IV, 2IV^*)$		D_4
(IV, IV^*)		D_4
$(2IV, I_2, I_6)$	\emptyset	$12A_1$
(IV, I_2, I_6)	\emptyset	$D_4, 8A_1$
$(2IV^*, I_1, I_3)$		$4A_1$
(IV^*, I_1, I_3)		$4A_1$
(I_9, I_1, I_1, I_1)	\emptyset	$12A_1$
(I_5, I_5, I_1, I_1)	\emptyset	$12A_1$
(I_3, I_3, I_3, I_3)	\emptyset	$12A_1$

TABLE 4. Singularities on the canonical cover of an Enriques surface with an extremal, special, elliptic fibration

fibration such that two components of the conductrix meet the fiber without being contained in it. This is not possible by Lemma 12.2. Hence, we have the conductrix in our table and the isolated singularities because of Lemma 12.7 (b).

- $(\text{II}^*, \text{I}_1)$: The conductrix is the one in the table by the same argument as in the (II^*) case. By Lemma 12.7 (b), we get the types of isolated singularities.
- $(\text{IV}, 2\text{IV}^*)$: Since N meets a simple component of the fiber of type IV^* , we can exclude the case where the conductrix does not have the full fiber as support, since in this case every simple component of the fiber of type IV^* has s -invariant 1 and r -invariant 0 while N has s -invariant 1, contradicting Lemma 12.7 (1). The isolated singularities are as in the table, since by [19] the fibers of type IV acquires a D_4 -singularity after Frobenius pullback.
- (IV, IV^*) : Suppose that A has the full fiber of type IV^* as support. Then N meets a multiplicity 2 component of this fiber, since $A \cdot N \leq 1$. But then N and components of the fiber of type IV^* form a fiber of type I_1^* of a different elliptic fibration such that two components of the conductrix meet the fiber without being contained in it. This is not possible by Lemma 12.2. As in the previous case, we get a D_4 -singularity.
- $(2\text{IV}, \text{I}_2, \text{I}_6)$ and $(\text{IV}, \text{I}_2, \text{I}_6)$: The argument is essentially the same as in the $(2\text{III}, \text{I}_8)$ and (III, I_8) cases.
- $(2\text{IV}^*, \text{I}_1, \text{I}_3)$ and $(\text{IV}^*, \text{I}_1, \text{I}_3)$: The argument is similar to the cases with singular fibers $(\text{IV}, 2\text{IV}^*)$ and (IV, IV^*) , except that the fibers of type I_n give 4 A_1 -singularities by Lemma 12.7 (a).
- All singular fibers multiplicative: In these cases, we get 12 A_1 -singularities by Lemma 12.7 (a). □

For the convenience of the reader, we give the corresponding table for quasi-elliptic fibrations. This does not require proof, since the conductrices are uniquely determined (see [7]) and the isolated singularities depend on the number of double fibers (see Lemma 12.5).

Lemma 12.11. *The isolated singularities on the normalization of the canonical cover of an Enriques surface with a quasi-elliptic fibration and the conductrix are summed up in table 5. The self-intersection number of the reduced inverse image of the curve on the minimal resolution of singularities of the canonical cover is given as an index to the multiplicity. We do not give multiplicities of the fibers of type III. The curve of cusps is encircled.*

Remark 12.12. Recall that any Enriques surface has a genus one fibration (Proposition 2.4) and if an Enriques surface X has a finite group of automorphisms, then any genus one fibration on X is extremal (Proposition 2.10). Therefore, X has an extremal, special genus one fibration by Proposition 2.11. Lemmas 12.10 and 12.11 imply that the canonical cover

Singular fibers	Conductrix	Isolated singularities
(2II^*)		$4A_1 \text{ or } D_4$
(II^*)		$4A_1 \text{ or } D_4$
(2I_4^*)		$4A_1 \text{ or } D_4$
(I_4^*)		$4A_1 \text{ or } D_4$
$(2\text{III}^*, \text{III})$		$4A_1 \text{ or } D_4$
$(\text{III}^*, \text{III})$		$4A_1 \text{ or } D_4$
$(2\text{I}_0^*, 2\text{I}_0^*)$		$4A_1$
$(2\text{I}_0^*, \text{I}_0^*)$		$4A_1 \text{ or } D_4$
$(\text{I}_0^*, \text{I}_0^*)$		$4A_1 \text{ or } D_4$
$(2\text{I}_2^*, \text{III}, \text{III})$		$4A_1 \text{ or } D_4$
$(\text{I}_2^*, \text{III}, \text{III})$		$4A_1 \text{ or } D_4$
$(2\text{I}_0^*, 4 \times \text{III})$		$4A_1 \text{ or } D_4$
$(\text{I}_0^*, 4 \times \text{III})$		$4A_1 \text{ or } D_4$
$(8 \times \text{III})$		$4A_1 \text{ or } D_4$


TABLE 5. Singularities on the canonical cover of an Enriques surface with a quasi-elliptic fibration

of any Enriques surface with finite automorphism group has only A_1 - or D_4 -singularities as isolated singularities.

12.3. Determination of possible dual graphs.

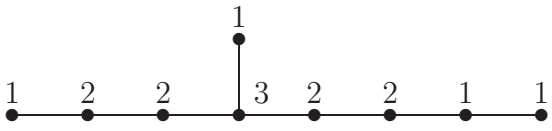
Theorem 12.13. *Assume that X is a classical or supersingular Enriques surface with a finite group of automorphisms. Then, the dual graph of (-2) -curves on X is one of the dual graphs given in Theorems 1.1 (A) and 1.2 (A).*

Proof. We start with a tuple (A, I) where A is one of the possible conductrices and I is either D_4 or $4A_1$. Recall that there exists a special genus one fibration on X (Proposition 2.11) which is extremal (Proposition 2.10). For a fixed (A, I) , we consider all possible special extremal genus one fibrations by using Lemmas 12.10 and 12.11, and check, if an Enriques surface with finite automorphism group with conductrix A and canonical double cover whose normalization has isolated singularities of type I can exist and determine its dual graph of (-2) -curves. We will make use of Lemma 12.2 very often without mentioning it from now on. Also we denote by N a special (-2) -section for a given special genus one fibration. If the fibration is quasi-elliptic, then N denotes the curve of cusps.

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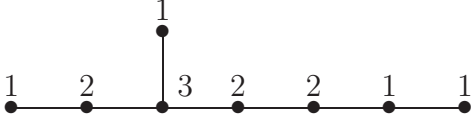
• Conductrix: D_4 or $4A_1$
 Singularities: D_4 or $4A_1$
 Possible special, extremal fibrations: (2II^*) quasi-elliptic

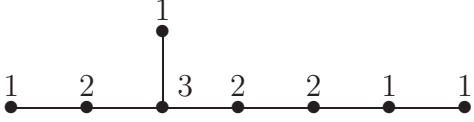
This is nothing but the dual graph of type \tilde{E}_8 . The Enriques surfaces are supersingular or classical according to the type of singularities (Corollary 12.6). These are the \tilde{E}_8 exceptional surfaces studied in [7].

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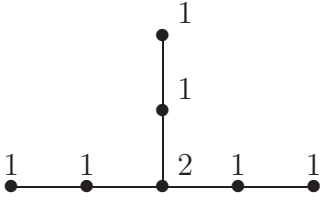
• Conductrix: D_4 or $4A_1$
 Singularities: D_4 or $4A_1$
 Possible special, extremal fibrations: (II^*) quasi-elliptic, $(2\text{III}^*, \text{III})$ quasi-elliptic and $(2\text{III}^*, 2\text{III})$ quasi-elliptic

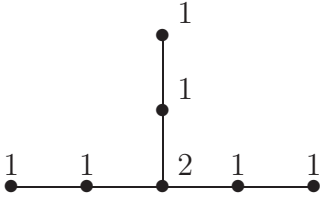
First note that in case of $(2\text{III}^*, \text{III})$ the 2-section N meets each component of the singular fiber of type III because otherwise there is a (-2) -curve meeting the conductrix more than once. Now, for each special genus one fibration we immediately obtain the dual graph of type $\tilde{E}_7 + \tilde{A}_1$. These are the \tilde{E}_7 exceptional surfaces of [7].

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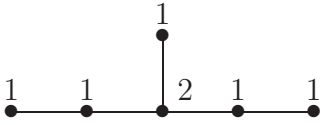
Conductrix: 
 Singularities: D_4 or $4A_1$
 Possible special, extremal fibrations: $(2I_4^*)$ quasi-elliptic, (II^*) elliptic and (II^*, I_1) elliptic

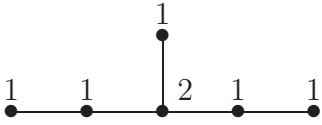
If we start with a special elliptic fibration with a singular fiber of type II^* , the 2-section N has to meet this fiber in a component with multiplicity 2, for otherwise there is a quasi-elliptic fibration with a double fiber of type $2III$. This is not allowed. Thus, we either get a quasi-elliptic fibration with a double fiber of type III^* or a quasi-elliptic fibration with a double fiber of type I_4^* . Again, the first case is not allowed. Therefore, this is an Enriques surface of type \tilde{D}_8 . Starting from the quasi-elliptic fibration of type $(2I_4^*)$, we immediately obtain the dual graph of type \tilde{D}_8 .

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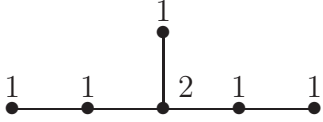
Conductrix: 
 Singularities: D_4 or $4A_1$
 Possible special extremal fibrations: $(III^*, 2III)$ quasi-elliptic, (III^*, III) quasi-elliptic, $(IV, 2IV^*)$ elliptic and $(2IV^*, I_3, I_1)$ elliptic

If we start with a special genus one fibration (III^*, III) together with the 2-section N , then we find a genus one fibration with a double fiber of type IV^* , and if we start with $(IV, 2IV^*)$ or $(2IV^*, I_3, I_1)$, then we find a fibration $(III^*, 2III)$. In the case of $(III^*, 2III)$, we immediately obtain the dual graph of type $\tilde{E}_6 + \tilde{A}_2$ (we can prove the non-existence of (III^*, III) quasi-elliptic case, but we omit the details). This is an \tilde{E}_6 exceptional Enriques surface of [7].

- 

Conductrix: 
 Singularities: D_4
 Possible special extremal fibrations: (I_4^*) quasi-elliptic, $(2I_2^*, III, III)$ quasi-elliptic, $(2I_2^*, 2III, III)$ quasi-elliptic

If we start with (I_4^*) , then we find a special fibration with a double fiber of type I_2^* . In cases $(2I_2^*, III, III)$ and $(2I_2^*, 2III, III)$, there exists a genus one fibration with a fiber of type III^* which is elliptic since the conductrix is contained in a fiber. Hence it is of type (III^*, I_2) which contradicts the type of singularities (Lemma 12.7, (b)). Thus, this case does not occur on an Enriques surface with finite automorphism group.

- Conductrix: 

Singularities: $4A_1$
 Possible special extremal fibrations: $(2I_2^*, 2III, III)$ quasi-elliptic, $(2I_2^*, III, III)$ quasi-elliptic, (I_4^*) quasi-elliptic and (III^*, I_2) elliptic

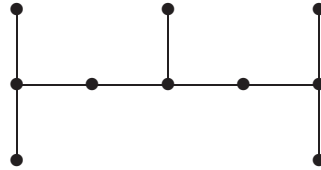
In every case, there is a quasi-elliptic fibration with a singular fiber of type (I_4^*) and with the curve of cusps meeting the central component.

To see this in the case of the special elliptic fibration with singular fibers of type (III^*, I_2) , note that if the 2-section meets a simple component of the fiber of type III^* , we get a quasi-elliptic fibration with a singular fiber of type $2III$, if it meets a component of multiplicity 2 on one of the long arms, we get a quasi-elliptic fibration with a singular fiber of type $2I_2^*$ and if it meets the component of multiplicity 2 in the center, there would be a special elliptic fibration with a double fiber of type IV^* , which we have excluded.

In the cases with a double fiber of type I_2^* , observe that the curve of cusps can not meet a component of a simple fiber of type III twice, because of Lemma 12.7 (2). Hence we obtain a quasi-elliptic fibration with a singular fiber of type I_4^* .

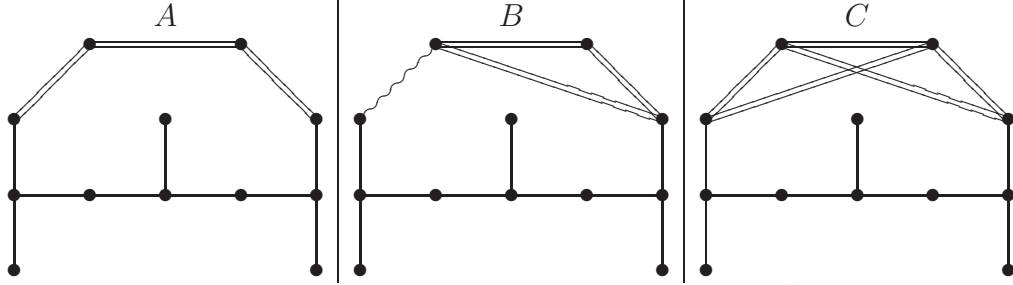
We will now start from a quasi-elliptic fibration with a singular fiber of type I_4^* and exclude this case. Two of the blown up points lie on the conductrix and two do not. Any (-2) -curve not meeting the conductrix has r -invariant 2 and therefore it passes through the 2 blown up points not lying on the conductrix. In particular, any two (-2) -curves not meeting the conductrix meet each other at least twice.

The configuration we start with is the following:

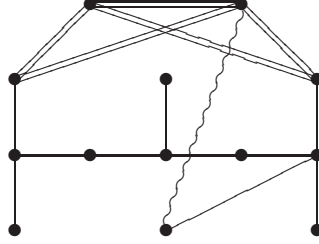


There are four subdiagrams of type \tilde{E}_7 . If the automorphism group of an Enriques surface with this conductrix is finite, the elliptic fibrations induced by those subdiagrams have singular fibers of type (III^*, I_2) . For any of these diagrams of

type \tilde{E}_7 , the two remaining curves are either 2- or 4-sections of the fibration, depending on whether the fiber of type III^* is double or not. If such a multisection meets a component of the fiber of type I_2 only once, we obtain a quasi-elliptic fibration with singular fiber of type II^* , which is not allowed. If one of the multisections meets only one component of the fiber of type I_2 , the other multisection and the other component of the fiber of type I_2 are disjoint from a diagram of type \tilde{D}_6 , hence they meet each other twice. This leaves us with the following three possible dual graphs, where a wiggly line means that the two curves corresponding to the adjacent vertices meet four times:

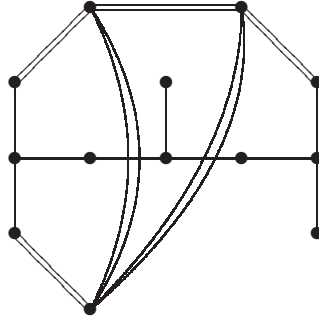


- We first exclude Case C . Using one of the diagrams of type \tilde{A}_1 , which yields a quasi-elliptic fibration with singular fibers $(2\text{I}_2^*, 2\text{III}, \text{III})$, we get the following graph:



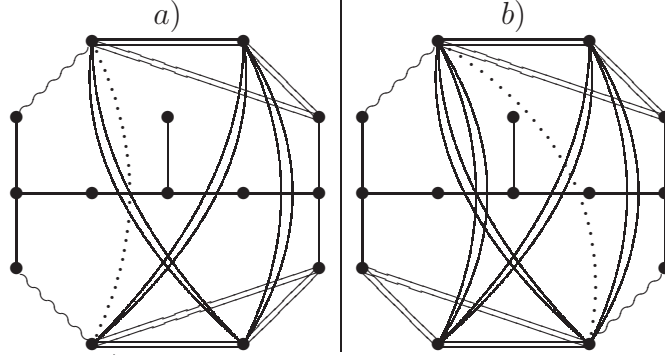
Therefore, there is a subdiagram of type \tilde{D}_4 . This is not allowed for an Enriques surface with finite automorphism group having this conductrix.

- Now we exclude Case A . We get another (-2) -curve as in the following diagram from one of the other fibrations with singular fibers of type $(\text{III}^*, \text{I}_2)$

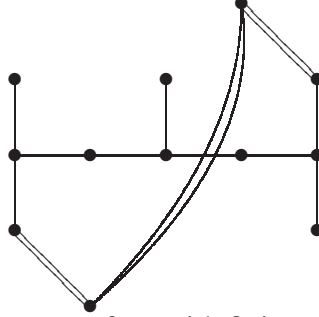


But then the orthogonal complement of a diagram of type \tilde{D}_6 contains a 2-connected path of four (-2) -curves, which is not possible.

- Lastly, let us exclude Case *B*. Again, looking at another fibration with singular fibers of type (III^*, I_2) , we get the following two cases, where a dotted line denotes that the two adjacent curves meet 10 times



In case *a*) we get the same contradiction as for Case *A*. In case *b*) there is a special elliptic fibration with singular fibers of type (III^*, I_2) having intersection graph of Case *A*, namely the following:



Therefore, an Enriques surface with finite automorphism group and this conductrix can not exist.

- Conductrix: $\bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet$

Singularities: $4A_1$

Possible special extremal fibrations: $(2I_0^*, 2I_0^*)$ quasi-elliptic and (I_4^*) elliptic

If we start with a special elliptic fibration with a singular fiber of type (I_4^*) , we have to observe that a special 2-section N has to meet the conductrix, for otherwise we obtain a quasi-elliptic fibration with a singular fiber of type 2III . Now, if the 2-section N meets the conductrix, we obtain a special genus one fibration with a singular fiber of type $2I_2^*$, $2I_1^*$ or $2I_0^*$. The first two are not allowed. Thus, we get

a quasi-elliptic fibration with a double fiber of type I_0^* and an Enriques surface of type $\tilde{D}_4 + \tilde{D}_4$.

The same graph is immediately obtained when starting with the quasi-elliptic fibration with singular fibers of type $(2I_0^*, 2I_0^*)$.

- Conductrix: $\overset{1}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{1}{\bullet}$
 Singularities: D_4 or $4A_1$
 Possible special extremal fibrations: $(2I_0^*, I_0^*)$ quasi-elliptic

Starting with a fibration with singular fibers of type $(2I_0^*, I_0^*)$, the special 2-section N meets the component with multiplicity 2 of the singular fiber of type I_0^* (otherwise there exists a fibration with a fiber of type III containing a component N of the conductrix), and hence there is a subdiagram of type \tilde{D}_7 which defines a non-extremal fibration (Propositions 2.7 and 2.8). Therefore, an Enriques surface with this conductrix can not have a finite automorphism group.

- Conductrix: $\overset{1}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{1}{\bullet}$
 Singularities: D_4 or $4A_1$
 Possible special extremal fibrations: (I_0^*, I_0^*) quasi-elliptic, $(2I_0^*, 2III, III, III, III)$ quasi-elliptic and $(2I_0^*, III, III, III, III)$ quasi-elliptic

Starting with a quasi-elliptic fibration with singular fibers of type (I_0^*, I_0^*) , we obtain an elliptic fibration with a singular fiber of type I_2^* , which is not allowed.

As for the fibrations with a double fiber of type $2I_0^*$, by the same reason as in the previous case, a special 2-section N meets two components of each simple fiber of type III. Therefore there is a diagram of type \tilde{D}_6 containing the conductrix. But an elliptic fibration with a fiber of type I_2^* can not be extremal by Propositions 2.7 and 2.8.

- Conductrix: $\overset{1}{\bullet} \text{---} \overset{1}{\bullet}$
 Singularities: D_4 or $4A_1$
 Possible special extremal fibrations: $(I_0^*, 2III, 2III, III, III)$ quasi-elliptic, $(I_0^*, 2III, III, III, III)$ quasi-elliptic, $(I_0^*, III, III, III, III)$ quasi-elliptic and (I_1^*, I_4) elliptic.

If there is a quasi-elliptic fibration on this surface, then there is a configuration of type I_0^* containing the conductrix. The induced elliptic fibration is not extremal.

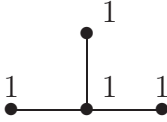
Starting with a special elliptic fibration with singular fibers of type (I_1^*, I_4) , we look at the intersection of N with the fiber of type I_1^* . If the special 2-section N meets distinct components, we obtain a configuration giving a double fiber of type I_4 or I_5 , which is a contradiction. If N meets a simple component twice, we get a double fiber of type III of a quasi-elliptic fibration and we have excluded this case before. If N meets a double component once, then there is a configuration of type I_0^* containing the conductrix giving the same contradiction as in the first paragraph.

- Conductrix: \bullet^1

Singularities: D_4 or $4A_1$

Possible special extremal fibrations: $(III, III, III, III, III, III, III, III)$ quasi-elliptic, any multiplicities

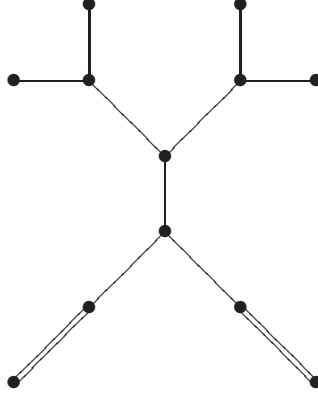
The 2-section N is nothing but the conductrix and hence N meets two components of each simple fiber of type III as in the previous cases. Thus we have an elliptic fibration with a fiber of type I_0^* which is not extremal by Proposition 2.7.

- Conductrix: 

Singularities: $4A_1$

Possible special extremal fibrations: $(I_2^*, 2III, 2III)$ quasi-elliptic, $(I_2^*, III, 2III)$ quasi-elliptic, (I_2^*, III, III) quasi-elliptic, $(2I_1^*, I_4)$ elliptic and (IV^*, I_1, I_3) elliptic

If there is a quasi-elliptic fibration with singular fibers of type $(I_2^*, 2III, 2III)$, we have the following configuration of (-2) -curves:

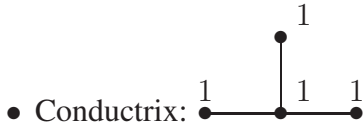


The special elliptic fibration induced by the diagram of type \tilde{D}_5 meeting the two curves at the bottom gives four more (-2) -curves. We leave it to the reader to check that the resulting intersection graph is of type VIII.

If there is a special elliptic fibration with singular fibers of type $(2I_1^*, I_4)$, the 2-section N has to meet a component of the fiber of type I_4 twice, since a special elliptic fibration with a double fiber of type IV is not allowed. Therefore, there is a quasi-elliptic fibration with a double singular fiber of type III, which has to be a fibration with singular fibers of type $(I_2^*, 2III, 2III)$, since the curve of cusps does not meet one of the components of the second fiber of type III which is a component of the fiber of type I_4 and the curve of cusps may not meet the other component twice.

Starting with a quasi-elliptic fibration with singular fibers of type $(I_2^*, III, 2III)$ or (I_2^*, III, III) , we immediately get the existence of a special elliptic fibration with a singular double fiber of type I_1^* , returning us to the case above.

If there is a special elliptic fibration with singular fibers of type (IV^*, I_1, I_3) , the 2-section meets either a simple component of the fiber of type IV^* twice or a double component once. In the first case, we get a quasi-elliptic fibration with a singular fiber of type $2III$ and in the second case, we get a special elliptic fibration with a double fiber of type I_1^* . Both cases have already been dealt with.



• Conductrix:

Singularities: D_4

Possible special extremal fibrations: $(I_2^*, III, 2III)$ quasi-elliptic, (I_2^*, III, III) quasi-elliptic and (IV, IV^*) elliptic

If we start with a quasi-elliptic fibration, we get a special elliptic fibration with a double fiber of type I_1^* , which is not allowed.

In the case of the fibration with singular fibers of type (IV, IV^*) , the 2-section either meets a simple component of the fiber of type IV^* twice, or a double component once. The first case leads to a special genus one fibration with a double fiber of type III and the second one to a special elliptic fibration with a double fiber of type I_1^* . Both cases have already been treated.

- Conductrix: \emptyset
Singularities: $D_4, 8A_1$
Possible special extremal fibrations: (IV, I_2, I_6) elliptic and (III, I_8) elliptic

We start from any of the two special fibrations and a special 2-section N . By considering the intersection of N with the fibers of type I_6, I_8 , we can find a special genus one fibration with an additive double fiber of type III or IV no matter how the 2-section intersects the fibers. However, these fibrations are not allowed by our list. Hence a surface with these singularities can not have finite automorphism group.

- Conductrix: \emptyset
Singularities: $12A_1$
Possible special extremal fibrations: (I_9, I_1, I_1, I_1) elliptic, (I_5, I_5, I_1, I_1) elliptic, $(2IV, I_2, I_6)$ elliptic, $(2III, I_8)$ elliptic and (I_3, I_3, I_3, I_3) elliptic

If we start with a special fibration with singular fibers of type $(2III, I_8)$, the 2-section has to meet two adjacent components of the fiber of type I_8 . Indeed, the twelve blowing-ups for the dissolution all happen on the singular fibers and the eight of them occurring on the fiber of type I_8 are the blowing-ups of the intersections of any two adjacent components. Since we have to blow up two points on the special 2-section, it has to meet such a point of intersection. From this configuration we leave it to the reader to verify, using the above list, that the dual graph we obtain is the one of type VII.

Starting with a special extremal fibration with singular fibers of type $(2IV, I_2, I_6)$, we can check that there is a special fibration with double fiber of type $2III$, which returns us to the case above. Indeed, if the 2-section meets distinct components of every fiber, we obtain a fibration with a singular fiber of type II^* which is not allowed by the assumption $A = \emptyset$.

For the other configurations, we also obtain a special elliptic fibration with a degenerate double fiber from the 2-section and components of the fiber of type I_n with $n \geq 3$. Hence, the argumentation of the previous two cases applies.

□

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